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Two Examples of Lecture Supplements: Exercise in Mathematical Economics

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ABSTRACT

This note shows by a couple of examples how to incorporate additional elements with conventional teaching materials in graduate microeconomics and macroeconomics. This note also provides examples of exercises that students might be assigned in class.

Keywords: *Convex sets, stationary process, covariance, Cauchy- Schwarz inequality, Gaussian process, Hilbert space.*

1. INTRODUCTION

Two examples, one from micro (the Profit function) and another from macro (second-order stationary process) are presented here. There is a hierarchy of results and most of the results are given in propositions and exercises. The profit function is cited here because it is an important building block of microeconomics. It provides flexibility to deal with the standard optimization models although the monotonicity assumption is invoked and here is the rub. In macroeconomics, too, it is stationary process, which though theoretically inconsistent with the assumption of stationarity, is used to incorporate into a time series empirical macro models. Loosely speaking, stationary process is one whose properties do not change over time. Ultimately remember, though, there is no single tool that applies to all occasions.

2. THE PROFIT FUNCTION IN MICROECONOMICS

First, the Profit function. The Profit function for a competitive firm is

$$\Pi\ (p) = \sup_{y \subset Y}\ \sum_i p_i y$$

This measures the maximum profit which the firm can earn given prices p and technology **Y.** To show that it is convex function of **p,** suppose that y_i maximizes profit at prices at $\mathbf{p_1}$and $\mathbf{y_2}$. Maximize profit at$\mathbf{p_2}$. For some $\alpha \subset [0, 1]$, let $\bar{p}$ be the weighted average price, that is,

$$\bar{\mathbf{p}} = \alpha\,\mathbf{p_1} + (1 - \alpha)\,\mathbf{p_2}$$

Now suppose that $\bar{\mathbf{y}}$ maximizes profits at $\bar{\mathbf{p}}$. Then

$$\Pi(\bar{p}) = p^T \bar{y} = (\alpha p_1 + (1-\alpha) p_2)^T \bar{y} = \alpha p_1^T \bar{y} + (1-\alpha) p_2^T$$

But since $y_1$and y_2 maximize profit at $\mathbf{p}_1$ and $\mathbf{p}_2$..

$$\alpha \mathbf{p}_1^{\mathbf{T}} \bar{\mathbf{y}} \le \alpha \mathbf{p}_1^{\mathbf{T}} \mathbf{y}_1 = \alpha\Pi(\mathbf{p}_1)$$

$$(1-\alpha)\,\mathbf{p}_2^{\mathbf{T}}\bar{\mathbf{y}} \le (1-\alpha)\,\mathbf{p}_1^{\mathbf{T}} = \alpha\Pi(\mathbf{p_2})$$

So

$$\Pi(\bar{p}) = \Pi(\alpha p_1 + (1-\alpha) p_2) = \alpha p_1^T \bar{y} + (1-\alpha) p_2^T \bar{y} \le \alpha\Pi(p_1) + (1-\alpha)\Pi(p_2)$$

This establishes that the profit function Π is convex in **p**

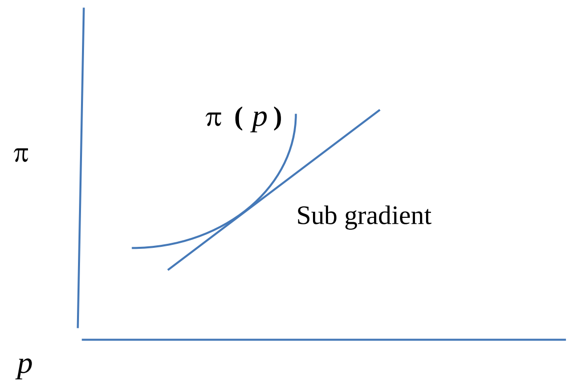

Figure shows a cross section of the profit function of a competitive firm. The straight line is the graph of a sub gradient (the derivatives of a convex function is a sub gradient) of the profit function. Let f be a function defined on a convex set S in a normed linear space X. For every $x_0 \in \text{int}\, S$ there exists a linear functional $\mathbf{g} \in X_0^*$ that bounds f in the sense that $f(x_0) \ge f(x_0) + g(x - x_0)$ for every. The fact that the profit function of a competitive firm is convex has an interesting ramification.[1] For example it implies that price

[1] The one ingredient of the Robinson Crusoe that cannot be dispensed with is convexity. Convexity of both technology and preference is indispensable to ensure the separation of production and upper preference sets and hence the possibility of decentralization through markets. With many agents the convexity requirements can be relaxed somewhere. On the production side, convexity of

stabilization will reduce average profits. Suppose that prices are random taking the values $(\mathbf{p}_1, \mathbf{p}_2 \ldots . \mathbf{p}_n)$ with probabilities $(\alpha_1, \ldots . \alpha_n)$. On average, the competitive firm will earn the expected profit

$$\Pi = \sum_{i=1}^{n} \alpha_i \Pi(\mathbf{p}_i)$$

Now suppose that prices are stabilized at the average price

$$\bar{\mathbf{p}} = \sum_{i=1}^{n} \alpha_i \mathbf{p}_i$$

Since the profit function is convex, Jensen's inequality implies that

$$\Pi(\mathbf{p}) \leq \overline{\Pi} = \sum_{i=1}^{n} \alpha_i \Pi(\mathbf{p}_i)$$

Price stabilization reduces expected profit. The intuition is straightforward. When the price is allowed to vary, the firm can tailor its production to the prevailing prices in each period. When the price is stabilized, the firm is not encouraged to respond optimally to price variations.

Remark 1

Economic life often takes place at the boundaries of convex sets, where the possibility of discontinuities must be taken into account. This accounts for some of the unwelcome contortions necessary in, for example, duality theory, which could otherwise be exhibited rather more elegantly. To define a discontinuous convex function a laMilgrom and Segal [7], let $S = \Re_{+.}$ the function f : s $\rightarrow \Re$ and

$$f(x) = \begin{bmatrix} 1 & x = 0 \\ 0 & 0\text{therwise} \end{bmatrix}$$

isconvex on S but discontinuous at 0.

Remark 2

A set may have convex cross sections without itself being convex. This is an important distinction in production theory. As an illustration, a banana produces convex slices but the banana itself is not a convex set. This example is cited by Carter [2]. Logarithmic transformations are often used in economic analysis. It is nice to know that they preserve concavity, since log is both concave and increasing.

Exercise 1

Let f be a convex function on an open set that is bounded above by ϖ in a neighborhood of x_0 ; that is, there exists an open set f containing x_0. Show that $f(x) \leq M$ for every $x \in \Theta$

Remark 3

Assuming f is non negative definite.
F concave $\Rightarrow$ log f concave
Log f convex $\Rightarrow$f convex

3. STATIONARY PROCESS IN THOMAS SARGENT'S TEXT–MACRECONOMIC THEORY

The condition for a process to be stationary is rather stringent. Thomas Sargent [9] in his text Macroeconomic Theory defines a process [x (t) , t > 0] to be a second order stationary or covariance stationary process if E [x (t)] = c and cov (x (t), x (t + s)) does not depend on t , that is, a process is second – order stationary (also called weakly stationary) if the first two moments of x (t) are the same for all t and the covariance between x (s) and x (t) depends only on t – s. From a second-order stationary process, let R(s) =cov (x (t), x (t + s)) be a finite dimensional distribution of a Gaussian process (being multivariate normal) are determined by their means and covariances, it follows that a second-order stationary process is stationary (see any standard text like Cox and Miller [4], Bhattacharya and Waymire[1][2]. However, there are many examples of second-order stationary process that are not stationary.

Example - A Moving Average Process

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the aggregate production set suffices, even if the technology of individual producers is non convex. (Hilden brand and Kirman [5].

[2] The main advantage of wide-sense stationarity is that it places the time-series in the context of Hilbert spaces. Let H be the Hilbert space generated by {x(t)}. By the positive definiteness of the auto covariance function, it follows from Bochner's theorem that there exists a positive measure μ on the real line such that H is isomorphic to the Hilbert subspace of $L^2(\mu)$ generated by $\{e^{-2\pi i\xi \cdot t}\}$. This, then, gives the following Fourier-type decomposition for continuous time stationary stochastic process: there exists a stochastic process $\omega\lambda$ with orthogonal increments such that, for all t

$$x(t) = \int e^{-2\pi i\lambda \cdot t} d\omega_\lambda,$$

where the integral on the right hand side is interpreted in a suitable (Riemann) sense. The Same result holds for a discrete-time stationary process, with the spectral measure now defined on the unit circle.

Let w_0, w_1, w_2 … be uncorrelated with $(E(x_n) = \mu \;\; var(w)_n = \sigma^2,\; n \geq 0$ and

for some positive integer k define

$$x_n = \frac{w_n + w_{n-1} + w_{n-2} + \dots \;\; w_{n-k}}{k+1}, \quad n \geq k$$

The process $[x_n, \quad n \geq k]$ which at each time keeps track of the arithmetic average of the most recent
k + 1 values of the w's is called a moving average process. Using the fact that $w_n, \geq 0$, we see that

$$Cov(x_n, x_{n+k}) = \begin{bmatrix} \frac{(k+1-m)\sigma^2}{k+1} & \text{if} \quad 0 \leq m \leq k \\ 0 & \text{if} \quad m \geq k \end{bmatrix}$$

Hence $[\; x_n, \quad n \geq k]$ is a second-order stationary process[3]. Let $[x_n, \; n \geq 1]$ be a second-order stationary process with $E[x_n] = \mu$. An important question when, if ever, does $\bar{x}_n = \sum_{i=1}^{n} x_i \; / n$ converges to μ. The following proposition shows that $E[\; \bar{x}_n - \mu)^2 \rightarrow 0$ if and only if $\sum_{i=1}^{n} R(i)/n \rightarrow 0$ That is, the expected square of the difference between $\bar{x}_n$ and μ will converge to 0 if and only if the limiting average value of $R(i)$ converges to0.

Theorem 1

Let $\{x_n, \quad n \geq 1\}$ be a second-order stationary process having mean μ and covariance function $R(i) = cov\{x_n, \; x_{n+1}\}$ and let $\bar{x}_n = \sum_{i=1}^{n} x/n$.

Then $\lim_{n \rightarrow \infty} E(x - \mu)^2] = 0$ if and only if

$$\lim \sum_{i=1}^{n} \frac{R(m)}{n} = 0$$

Proof

Let $y_i = x_i - \mu$ sand $\bar{y}_n = \sum_{i=1}^{n} y_i / n$ .Suppose that $\sum_{i=1}^{n} R(i)/n \rightarrow 0$. We need to show that this implies that $E(\bar{y}_n) \rightarrow 0$. Now,

$$E[\bar{y}_n^2] = \frac{1}{n^2} E\left[\sum_{i=1}^{n} y^2{}_i + 2 \sum_i \sum_{j \leq 1} y_i y_j\right]$$

$$= \frac{R(0)}{n} + \frac{2 \sum_i \sum_j R(j-i)}{n^2}$$

The RHD of the above goes to 0 when $\sum_i^n (i)/n \rightarrow 0$

To go the other way, suppose that $E[\bar{y}_n^2] \rightarrow 0$ then

$$\left(\sum_{i=0}^{n-1} \frac{R(i)}{n}\right)^2 = \left[\frac{1}{n} \sum_{i=1}^{n} cov(y_1, \;\; y_n\right]^2$$

$$= [cov(y_1, \bar{y}_n)]^2$$
$$= [E(y_1 \bar{y}_n)]^2$$
$$\leq E[y^2{}_1] \; E(\bar{y}_n^2]$$

which shows that $\sum_{i=0}^{n-1} R(i)/n$ as $n \rightarrow \infty$. The reader should note that we have made use of the Cauchy-Schwarz inequality, which states that for random variables x and y

$$(E[xy])^2 \leq E[x^2] E[y^2]$$

Remark 4

If the second moment of x_t is finite for all t then the mean is $E(x_t)$,the variance $var(x_t) = E[(x_t - E(x_t))^2]$ and the covariance , $cov(x_{t_1}, x_{t_2})) =$

$$E[[x_t - E(x_{t_1}))(x_{t_2} - E(x_{t_2}))] \quad \text{finite for all} \quad t, t_1 \; t_2$$

Exercise 2

Refer to remark 4. Show why?
Hint: Use the Cauchy-Schwarz inequality

Exercise 3

[3] The terminology used for types of stationarity other than strict stationarity can be rather mixed. For example, Priestley [8] uses stationary up to order*m* if conditions similar to those given here for wide sense stationarity apply relating to moments up to order *m*. Thus wide-sense stationarity would be equivalent to "stationary to order 2".Honarkhah and Caers [6] uses the assumption of stationarity in the context of multiple-point geostatisticswhere higher n-point statistics are assumed to be stationary in the spatial domain.

Which of the following function could serve as the covariance function for a stationary process?

(a) $\phi_1(\Gamma) = e^{|\Gamma|} \quad \Gamma \in \Re$

(b) $\phi_2(\Gamma) = e^{-\Gamma^2} \quad \Gamma \in \Re$

(c) $\phi_3(\Gamma) = e^{-\Gamma} \cos \Gamma_3 \quad \Gamma \in \Re$

(d) $\phi_1(\Gamma) = e^{-\Gamma^2} \sin \Gamma \quad \Gamma \in \Re$

4. CONCLUSION

This note made an attempt to portray a framework for adding materials into micro and macroeconomics teaching at the graduate level. While I hope this incorporation might be interesting, to make the most of it would require some effort on students' part. Remember one of the most important requirements for understanding to economics, especially mathematical economics, is to build up an appropriate mental framework. Remember that a good economic model strips away all the unnecessary and distracting detail and focuses attention on the essentials of a problem or issue. Abstraction reveals the logical structure of a mathematical framework in the same way as it reveals the logical structure of an economic model.

COMPETING INTERESTS

Authors have declared that no competing interests exist.

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