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Invited By The Mersenne Primes, The Perfect Numbers And The Mersenne Composite Numbers.

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***Abstract*—In this paper, via divisibility, we show a simple Theorem which helps to characterize the Mersenne primes, the even perfect numbers and the Mersenne composite numbers. We recall that a *Mersenne prime* (see [1] or [4] or [5] or [6] or [7]) is a prime of the form $M_m = 2^m - 1$, where m is prime; for example M_{13} and M_{19} are *Mersenne prime*; and Mersenne primes are known for some integers $> M_{19}$. A *Mersenne composite number* or a *Mersenne composite* (see [2] or [3]) is a non prime number of the form $M_m = 2^m - 1$, where m is prime; it is known that M_{11} and M_{67} are Mersenne composite; and Mersenne composite are known for some integers $> M_{67}$. Finally, we recall that Pythagoras saw *perfection* in any integer that equaled the sum of all the other integers that divided evenly into it (see [2]). The first perfect number is 6. It's evenly divisible by 1, 2, and 3, and it's also the sum of 1, 2, and 3, [note 28, 496 and 33550336 are also perfect numbers (see [2])]; and perfect numbers are known for some integers > 33550336.**

Keywords: *Mersenne primes, Mersenne composite, perfect numbers.*
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Preliminary. This paper is divided into two sections. In section.**1**, we state and prove a Theorem which implies the characterizations of the Mersenne primes, the even perfect numbers and the Mersenne composite numbers. In section.**2**, using the Theorem of section.**1**, we characterize the Mersenne primes, the even perfect numbers and the Mersenne composite numbers.

1. Statement and the proof of Theorem which implies the characterizations of the Mersenne primes, the even perfect numbers and the Mersenne composite numbers.

We recall that for every integer $n \geq 1$, $n!$ is defined as follow:

$$n! = \begin{cases} 1 & \text{if } n = 1, \\ 2 & \text{if } n = 2, \\ 1 \times 2 \times .... \times n & \text{if } n \geq 3. \end{cases}$$

Theorem 1.1. *Let m be an integer ≥ 3, and look at $2^m - 1$. Then, $2^m - 1$ is prime or $2^m - 1$ divides $(2^m - 2)!$.*

Before proving Theorem 1.1, let us remark the following.

Remark 1.2. *Let m be an integer ≥ 3; and put $n = 2^m - 2$. If $n + 1 = p^2$ (where p is prime), then $n + 1$ divides $n!$.*

Proof. Otherwise [we reason by reduction to absurd], clearly

$$p^2 \text{ does not divide } n! \qquad (\mathbf{1.2.0}),$$

and we observe the following.

Observation 1.2.1. *p is a prime ≥ 3.*

Otherwise, clearly $p = 2$, and noticing (via the hypotheses) that $n + 1 = p^2$, then using the previous two equalities, it becomes trivial to deduce that $n = 3$; a contradictionn, since $n > 4$ (via the hypotheses).

Observation 1.2.2. *$p \leq n$.*

Otherwise, $p > n$ and the previous inequality clearly says that

$$p \geq n + 1 \qquad (1.2.2.0).$$

Now noticing (via the hypotheses) that $n + 1 = p^2$, then, using the previous equality and using *(1.2.2.0)*, we trivially deduce that

$$p \geq n + 1 \text{ and } n + 1 = p^2 \qquad (1.2.2.1).$$

(1.2.2.1) clearly says that $p \geq p^2$; a contradiction, since $p \geq 3$ (by using Observation 1.2.1).

Observation 1.2.3. *$2p \leq n$.*

Otherwise, $2p > n$ and the previous inequality clearly says that

$$2p \geq n + 1 \qquad (1.2.3.0)$$

Now noticing (via the hypotheses) that $n + 1 = p^2$, then, using the previous equality and using *(1.2.3.0)*, we trivially deduce that

$$2p \geq n + 1 \text{ and } n + 1 = p^2 \qquad (1.2.3.1).$$

(1.2.3.1) clearly says that $2p \geq p^2$; a contradiction, since $p \geq 3$ (by using Observation 1.2.1). Observation 1.2.3 follows.

Observation 1.2.4. *$2p \neq p$.*

Indeed, it is immediate that $2p \neq p$, since $p \geq 3$ (by using Observation 1.2.1). Observation 1.2.4 follows.

The previous trivial observations made, look at p (recall that p is prime); observing (by Observations 1.2.2 and 1.2.3 and 1.2.4) that $p \leq n$ and $2p \leq n$ and $p \neq 2p$, then, it becomes trivial to deduce that

$$\{p, 2p\} \subseteq \{1, 2, 3, ..., n - 1, n\} \qquad (1.2.5).$$

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(1.2.5) immediately implies that

$$p \times 2p \;\; divides \;\; 1 \times 2 \times 3 \times ... \times n-1 \times n \qquad (1.2.6).$$

(1.2.6) clearly says that $2p^2$ divides $n!$; in particular p^2 clearly divides $n!$ and this contradicts **(1.2.0)**. Remark 1.2 follows. □

The previous remark made, now we prove Theorem 1.1.

Proof of Theorem 1.1. Put $n = 2^m - 2$ and look at $n+1$. **If** $n+1$ is prime, then the proof is ended. **If** $n+1$ is not prime, then $n+1$ divides $n!$. Otherwise (we reason by reduction to absurd)

$$n+1 \, is \, not \; prime \; and \; n+1 \; does \; not \; divide \; n! \quad (\mathbf{1.1.0}),$$

and we observe the following.

Observation 1.1.1. *Let p be a prime such that $n+1$ is divisible by p (such a p clearly exists). Then $\frac{n+1}{p}$ is an integer* **and** $\frac{n+1}{p} \leq n$ **and** $p \leq n$ **and** $\frac{n+1}{p} = p$.

Indeed, it is immediate that $\frac{n+1}{p}$ is an integer [since p divides $n+1$], **and** it is also immediate that $\frac{n+1}{p} \leq n$ [otherwise, $n+1 > np$; now, remarking that $p \geq 2$ (since p is prime), then the previous two inequalities imply that $n+1 > 2n$; so $1 > n$ and we have a contradiction, since $n > 4$, by the hypotheses]. **Clearly** $p \leq n$ [otherwise, $p > n$; now, recalling that $n+1$ is divisible by p, then the previous inequality implies that $n+1 = p$. Recalling that p is prime, then the previous equality clearly says that $n+1$ is prime and this contradicts **(1.1.0)**]. That being so, to prove Observation 1.1.1, it suffices to prove that $\frac{n+1}{p} = p$. **Fact:** $\frac{n+1}{p} = p$ [otherwise, clearly $\frac{n+1}{p} \neq p$; now, remarking (by using the previous) that $\frac{n+1}{p}$ is an integer **and** $\frac{n+1}{p} \leq n$ **and** $p \leq n$; then it becomes trivial to deduce that $\frac{n+1}{p}$ **and** p are two different integers such that $\{p, \frac{n+1}{p}\} \subseteq \{1, 2, 3, ..., n-1, n\}$. The previous inclusion immediately implies that $p \times \frac{n+1}{p}$ divides $1 \times 2 \times 3 \times ... \times n-1 \times n$; therefore $n+1$ divides $n!$, and this contradicts **(1.1.0)**. So $\frac{n+1}{p} = p$]. Observation 1.1.1 follows.

The previous trivial observation made, look at $n+1$; observing (by using Observation 1.1.1) that p is prime such that $\frac{n+1}{p} = p$, clearly

$$n+1 = p^2, \; where \; p \; is \; prime \qquad (1.1.2).$$

Now using *(1.1.2)* and Remark 1.2, then it becomes trivial to deduce that $n+1$ divides $n!$, and this contradicts **(1.1.0)**. Theorem 1.1 follows. □

Theorem 1.1 immediately implies the characterizations of Mersenne primes, even perfect numbers and Mersenne composite numbers.

2. Characterizations of Mersenne primes, even perfect numbers and Mersenne composite numbers.

In this section, using Theorem 1.1, we characterize Mersenne primes, even perfect numbers and Mersenne composite numbers.

Theorem 2.1. *(Characterization of Mersenne primes). Let m be an integer ≥ 3 and look at $2^m - 1$. Then the following are equivalent.*
(1). $2^m - 1$ *is a Mersenne prime.*
(2). $2^m - 1$ *does not divide* $(2^m - 2)!$.

To prove Theorem 2.1, we need a Theorem of Euclide.

Theorem 2.2 *(Euclide). Let a, b and c, be integers such that $a \geq 1$, $b \geq 1$ and $c \geq 1$. If a divides bc and if the greatest common divisor of a and b is 1, then a divides c.*□

Corollary 2.3. *Let n be an integer ≥ 1 and look at $n!$. Now let p be a prime $\geq n+1$; then the greatest common divisor of $n!$ and p is 1 (in particular, p does not divide $n!$).*
Proof. Immediate, and follows immediately by using Theorem 2.2 and the definition of $n!$, and by observing that p is a prime $\geq n+1$.□

Now, we simply prove Theorem 2.1.

Proof of Theorem 2.1. (1) $\Rightarrow$ (2)]. Immediate, by puting $n = 2^m - 2$ and by remarking that $n+1$ is prime and by using Corollary 2.3.
(2) $\Rightarrow$ (1)]. Immediate. Indeed, since $2^m - 1$ does not divide $(2^m - 2)!$, then, using Theorem 1.1, we immediately deduce that

$$2^m - 1 \; is \; prime \qquad (2.1.0).$$

That being so, since $2^m - 1$ is prime (by *(2.1.0)*), then it becomes immediate to deduce that

$$m \; is \; prime \qquad (2.1.1).$$

Consequently $2^m - 1$ is a Mersenne prime (by using *(2.1.0)* and *(2.1.1)*). □

Using Theorem 2.1, then the following two Theorems become immediate.

Theorem 2.4. *(Characterization of even perfect numbers). Let m be an integer ≥ 3 and look at $2^m - 1$. Then the following are equivalent.*
(1). $2^{m-1}(2^m - 1)$ *is an even perfect number.*
(2). $2^m - 1$ *does not divide* $(2^m - 2)!$.
To prove Theorem 2.4, we need the following known Theorem

Theorem 2.5. *(Euler). The following are equivalent.*
(1). e *is an even perfect number.*
(2). e *is of the form* $2^{m-1}(2^m - 1)$*, where* $2^m - 1$ *is prime.*
□

Proof of Theorem 2.4. Immediate and follows by using Theorem 2.5 and Theorem 2.1. □

Theorem 2.6. *(Characterization of Mersenne composite numbers). Let m be a prime ≥ 3 and look at $2^m - 1$. Then the following are equivalent.*
(1). $2^m - 1$ *is a Mersenne composite number.*
(2). $2^m - 1$ *divides* $(2^m - 2)!$.
Proof. Immediate, by observing that m is a prime ≥ 3 and by using Theorem 2.1. □

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