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Distributed Control of N×N Elliptic System under Conjugation Conditions and Mixed Boundary Conditions

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ABSTRACT

In this study, we obtain some necessary (sufficient) conditions for non-cooperative n×n elliptic systems Optimal control problems for distributed parameter systems whose state are described mixed boundary-value problems with conjugation conditions are constructed. First, we prove the existence and uniqueness of solutions for these Systems and then we find the necessary and sufficient conditions for optimality for these systems. The main advantage offered by using the conjugation conditions.

Keywords: *Linear systems, Existence and uniqueness of solutions, n×n non cooperative system, mixed boundary condition, optimal control*

1. INTRODUCTION

There are many problems discussing the distributed control of systems governed by partial differential equations appeared, for example for one or two equations in the definite book on J.L. Lions [7], in the works of I.M. Gali [3-4] and in [5-6].Optimal control problems for non-cooperative elliptic systems involving Laplace operator discussed in [2], [1], Here we concerned with optimal control of boundary systems, generalize the discussion to n×n systems under conjugation conditions .We prove the existence of 2×2 elliptic systems with conjugation condition and mixed boundary conditions; then we find the set of equations and inequalities that characterize the optimal control of the boundary type of this system, generalize in n×n system. In [2] studied control problem with the scalar case for the state of the system. Conjugation conditions associated with very useful, for example, there are cracks in the earth's crust need to reduce these conditions lead to this reduction, it serves as water and sand overlying the ground to reduce these cracks. We have organized the paper as follows: in section 2, we study the case of 2×2 elliptic system then generalized the study to n×n; in section 3, we have presented the conclusion and future works to be done.

2. THE CASE OF 2×2 ELLIPTIC SYSTEMS WITH CONJUGATION CONDITION AND MIXED BOUNDARY CONDITION

In this section, we consider the following 2×2 elliptic system

$$\begin{cases} -\Delta y_1 + a y_1 - b y_2 = f_1 \ \text{in}\ \Omega_1 \\ -\Delta y_2 + b y_1 + a y_2 = f_2 \ \text{in}\ \Omega_2 \\ y_1 = 0,\ y_2 = 0,\ \frac{\partial y_1}{\partial v_A} = g_1, \frac{\partial y_2}{\partial v_A} = g_2 \ \text{on}\ \Gamma \end{cases} \qquad (1)$$

Where $\frac{\partial Y}{\partial v_A} = \left[\frac{\partial y_1}{\partial x_k}\cos(v,x_k) + \frac{\partial y_1}{\partial x_k}\cos(v,x_k)\right]$

on Γ , $\cos(v,x_k)$ =K−the direction cosine of v,

v being the normal at Γ exterior to Ω and $g_1, g_2,$

$\in H^{\frac{1}{2}}(\Gamma), k = 1,2, Y = \{y_1, y_2\}$

, since $[H_0^1(\Omega)]^2 \subseteq [H^1(\Omega)]^2$ then

$\|Y\|^2_{[H_0^1(\Omega)]^2} \leq \|Y\|^2_{[H^1(\Omega)]^2}$,a>0,a,b∈ ℝ and

conjugation conditions(*)

$\left[\frac{\partial y_1}{\partial v_A}\right] = \left[\frac{\partial y_1}{\partial x_1}\cos(v,x_k) + \frac{\partial y_1}{\partial x_2}\cos(v,x_k)\right] = 0$

$\left\{\frac{\partial y_1}{\partial x_1}\cos(v,x_k) + \frac{\partial y_1}{\partial x_2}\cos(v,x_k)\right\}^{\pm} = r[y_1]$

$\left[\frac{\partial y_2}{\partial v_A}\right] = \left[\frac{\partial y_2}{\partial x_1}\cos(v,x_k) + \frac{\partial y_2}{\partial x_2}\cos(v,x_k)\right] = 0$

$\left\{\frac{\partial y_2}{\partial x_1}\cos(v,x_k) + \frac{\partial y_2}{\partial x_2}\cos(v,x_k)\right\}^{\pm} = r[y_2]$, where Ω is an open subset of R^n with smooth boundary Γ,(−Δ) is

Laplace operator
$,\Omega = \Omega_1 \cup \Omega_2 \quad ,k = 1,2, \ \Gamma = (\partial\Omega_1 \cap \partial\Omega_2)\backslash\gamma,$

$\gamma = \partial\Omega_1 \cup \partial\Omega_2 \neq \emptyset$, $\cos(v,x_k)$ =K−the direction cosine of v, v being the normal at Γ

$,[\varphi] = \varphi^+ - \varphi^-$ $,\{\varphi(\varepsilon)\}^+ = \varphi(\varepsilon + \varepsilon)$ and

$\{\varphi(\varepsilon)\}^- = \varphi(\varepsilon - \varepsilon)$.

The given elliptic equation is specified in bounded, continuous and strictly Lipschitz domains in $\Omega_1, \Omega_2 \in R^n$.

2.1 Existence and uniqueness of Solution

Since $H_0^1(\Omega) \subseteq L^2(\Omega) \subseteq H^{-1}(\Omega)$,

Then we have chain of the form

$$[H_0^1(\Omega)]^2 \subseteq [L^2(\Omega)]^2 \subseteq [H^{-1}(\Omega)]^2$$

On Sobolev space $[H_0^1(\Omega)]^2 \times [H_0^1(\Omega)]^2$
$a(y, \Psi)=$

$$-\int_\Omega \{(\Delta y_1)\Psi_1 + (\Delta y_2)\Psi_2\}\, dx$$

$$+\int_\gamma \{r[y_1][\Psi_1] + r[y_2][\Psi_2]\, d\gamma +$$

$$\int_\Omega (ay_1\Psi_1 + a\, y_2\Psi_2 + b\, y_1\Psi_2 - b\, y_2\Psi_1)dx$$

$$L(\Psi)=\int_\Omega (f_1\Psi_1 + f_2\Psi_2)dx + \int_\Gamma (g_1\Psi_1 + g_2\Psi_2)\, d\Gamma$$

For $y=(y_1, y_2) \in [H_0^1(\Omega)]^2$ and $\Psi = (\Psi_1, \Psi_2) \in [H_0^1(\Omega)]^2$ and by lax- Milgram lemma, we prove that:

THEOREM 1: For $F = (f_1, f_2) \in [L^2(\Omega)]^2$, there exists a unique solution of $Y \in [H_0^1(\Omega)]^2$ system (1).

PROOF: The bilinear form can be written as $a(y, \Psi) = -\int_\Omega \{(\Delta y_1)\Psi_1 + (\Delta y_2)\Psi_2\}\, dx + \int_\gamma \{r[y_1][\Psi_1] + r[y_2][\Psi_2] d\gamma + \int_\Omega (a\, y_1\Psi_1 + a\, y_2\Psi_2 + b\, y_1\Psi_2 - b\, y_2\Psi_1)dx + \int_\Gamma \{F(y_2, \Psi_2) + F(y_1, \Psi_1)\} d\Gamma =$

$$-\int_\Omega \{(\Delta y_1)\Psi_1 + (\Delta y_2)\Psi_2\}\, dx$$

$$+\int_\gamma \{r[y_1][\Psi_1] + r[y_2][\Psi_2] d\gamma +$$

$$\int_\Omega (a\, y_1\Psi_1 + a\, y_2\Psi_2 + b\, y_1\Psi_2 - b\, y_2\Psi_1)dx$$

$$+\int_\Gamma \left(\Psi_1 \frac{\partial y_1}{\partial v_A} + \Psi_2 \frac{\partial y_2}{\partial v_A}\right) d\Gamma =$$

So

$$a(y, y) \geq \int_\Omega |\nabla y_1|^2 + |\nabla y_2|^2\, dx +$$

$$\int_\gamma \{r[y_1]^2 + r[y_2]^2 d\gamma + \int_\Omega ay_1^2 + ay_2^2\, dx$$

From Friedrich's inequality

$$\int_\Omega \left\{\left(\frac{\partial y_1}{\partial x_1}\right)^2 + \left(\frac{\partial y_2}{\partial x_2}\right)^2\right\} dx \geq \mu \int_\Omega \{y_1^2 + y_1^2\}\, dx$$

$\forall y_1 \in H_0^1(\Omega)$, $\mu = const > 0$ and embedding theorems [9], or from definition of norm on $H_0^1(\Omega)$

$$\int_\Omega |\nabla y_1|^2 + |\nabla y_2|^2\, dx + \int_\Omega ay_1^2 + ay_2^2\, dx$$

$$\geq \int_\Omega |\nabla y_1|^2 + |\nabla y_2|^2\, dx + \int_\Omega y_1^2 + y_2^2\, dx$$

$, r[y_1]^2, r[y_2]^2 \geq 0,$

$$a(y, y) \geq \|y\|^2_{(H_0^1(\Omega))^2} \geq \|y\|^2_{[H^1(\Omega)]^2} \tag{2}$$

[Coerciveness] $\forall y \in H_0^1(\Omega)$, c = const, $y = (y_1, y_2)$ $|a(y, \Psi)| \leq c_1 \|\Psi\|_{H_0^1(\Omega)} \|y\|_{H_0^1(\Omega)}$, $c_1 = const$, $\forall\, y, \Psi \in H_0^1(\Omega)$

and $L(\Psi)$ is continuous on $[H_0^1(\Omega)]^2$ then by Lax-Milgram lemma , there exist a unique solution $y=(y_1, y_2) \in H_0^1(\Omega)$, such that $a(y, \Psi) = L(\Psi)$ for all $\Psi \in [H_0^1(\Omega)]^2$.

2.2 Formulation of the control problem.

Let $[L^2(\Omega)] \times [L^2(\Omega)]$ being the space of controls.
The energy functional

$$\varphi(v) = a(y, v) - 2L(v) \tag{3}$$

A unique state y (u)$\in \{v|\Omega_i \in W_2^1(\Omega_i); i = 1,2; v|\Gamma = 0\}$, where $W_2^1(\Omega)$ is a set of the sobolev functions are specified on domain Ω_1 to every control u$\in [L^2(\Omega)]^2$. For a control u= $u = (u_1, u_2) \in [L^2(\Omega)]^2$ the state of the system y (u) = $\{y_1(u), y_2(u)\}$ is given by the solution of the following system:-

$$\begin{cases} -\Delta y_1 + ay_1 - by_2 = f_1 + u_1 \text{ in } \Omega_1 \\ -\Delta y_2 + by_1 + ay_2 = f_2 + u_2 \text{ in } \Omega_2 \\ y_1(u) = 0\,, y_2(u) = 0\,, \frac{\partial y_1}{\partial v_A} = g_1, \frac{\partial y_2}{\partial v_A} = g_2 \text{ on } \Gamma \\ \text{conditions}(*) \end{cases} \tag{4}$$

The observation equation is given by

z (u) $=\{z_1(u), z_2(u)\} = y(u)$.

The cost functional is given by:

$$J(u) = (\bar{a}u, u)_{[L^2(\Omega)]^2} + \|y(u) - z_d\|^2_\Omega, \tag{5}$$

$$= \int_\Omega (y_1(u) - z_{1d})^2\, dx + \int_\Omega (y_2(u) - z_{2d})^2\, dx + (\bar{a}u, u)_{[L^2(\Omega)]^2}$$

Where $Z_d = \{Z_{1d}, Z_{2d}\}$ in $[L^2(\Omega)] \times [L^2(\Omega)]$

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$, H \subset L^2(\Omega)$

$, \bar{a}$ is hermitian positive definite operator defined on $[L^2(\Omega)]^2$ such that:

$$(\bar{a}u, u) \geq \gamma \|u\|^2_{[L^2(\Gamma)]^2}, \gamma > 0 \qquad (6)$$

The control problem then is to find:

$$\begin{cases} u \in U_{ad} \text{ such that} \\ J(u) = \inf J(v) \text{ for all } v \in U_{ad} \end{cases}$$

,

where U_{ad} is a closed convex subset from $[L^2(\Omega)]^2$, since the cost function (3) can be written as

$$J(v) = \int_\Omega (y_1(v) - y_1(0)) + (y_1(0) - z_{1d}))^2 \, dx$$

$$+ \int_\Omega (y_1(v) - y_1(0)) + (y_1(0) - z_{1d}))^2 \, dx$$

$$+ (\bar{a}v, v)_{[L^2(\Omega)]^2}$$

$= \pi(v, v) - 2L(v) + \|z_{id} - y_i(0)\|^2$ i=1, 2, where

$$\pi(u, v) = (\bar{a}u, v)_{[L^2(\Omega)]^2} + (y_1(u) - y_1(0), y_1(v) - y_1(0))_{L_2(\Omega)} + (y_2(u) - y_2(0), y_2(v) - y_2(0))_{L_2(\Omega)},$$

$$L(v) = (z_{1d} - y_1(0), y_1(v) - y_1(0))_{L_2(\Omega)}$$

, since the difference $y_1(v) - y_1(0) = \tilde{y}_1(v)$ is the unique solution to problem (2).

It is necessary to assume $f = 0$ for them $\tilde{y}(\alpha_1 u_1 + \alpha_2 u_2) = \alpha_1 \tilde{y}(u_1) + \alpha_2 \tilde{y}(u_2)$ For all $\alpha_1, \alpha_2 \in R^1, u_1, u_2 \in L_2(\Omega)$ $, (\bar{a}u, v) = (\sqrt{\bar{a}}u, \sqrt{\bar{a}}v)$ substitute in bilinear form $\pi(u, u) \geq a_0(u, u)$, let $\tilde{y}' = \tilde{y}(u')$ and $\tilde{y}'' = \tilde{y}(u'')$ be solutions to (3), $u = u(x)$ then, $\pi(u, u)$ is coercive on $[L^2(\Omega)]^2$

,since $\|\tilde{y}' - \tilde{y}''\|^2 \leq \|\tilde{y}' - \tilde{y}''\|^2_v \leq \mu a(\tilde{y}' - \tilde{y}'', \tilde{y}' - \tilde{y}'')$, since $L(v)$ is continuous on $[L^2(\Omega)] \times [L^2(\Omega)]$ then there exist a unique optimal control $u \in U_{ad}$ [1], by another method since the cost functional is written by $J(v) = \pi(v, v) - 2L(v)$ and bilinear form coercive and linear form is continuous by theorem 1 in lions [1] then there exist a unique optimal control $u \in U_{ad}$. The function y is specified, minimizes the energy functional. Moreover, we have the following theorem which gives the characterization of the optimal control u:

THEOREM 2:

Assume that (2), (6) hold. The cost function being given (5), necessary and sufficient for u to be an optimal control is that the following equations and inequalities be satisfied

$$\begin{cases} -\Delta p_1 + ap_1 + bp_2 = y_1(u) - z_{1d} \text{ in } \Omega_1 \\ -\Delta p_2 - bp_1 + ap_2 = y_2(u) - z_{2d} \text{ in } \Omega_2 \\ p_1(u) = 0, p_2(u) = 0, \frac{\partial p_1}{\partial v_{A^*}} = \frac{\partial p_2}{\partial v_{A^*}} = 0 \quad , x \in \Gamma \\ \left[\frac{\partial p_1}{\partial x_1} \cos(v, x_k) + \frac{\partial p_2}{\partial x_2} \cos(v, x_k)\right] = 0, x \in \gamma \\ \left[\frac{\partial p_2}{\partial x_1} \cos(v, x_k) + \frac{\partial p_2}{\partial x_2} \cos(v, x_k)\right] = 0, x \in \gamma \\ \left\{\frac{\partial p_1}{\partial x_1} \cos(v, x_k) + \frac{\partial p_1}{\partial x_2} \cos(v, x_k)\right\}^{\pm} = r[p_1], x \in \gamma \\ \left\{\frac{\partial p_2}{\partial x_1} \cos(v, x_k) + \frac{\partial p_2}{\partial x_2} \cos(v, x_k)\right\}^{\pm} = r[p_2], x \in \gamma \end{cases} \qquad (7)$$

, p (u) $= \{p_1(u), p_2(u)\}$ is the adjoint state.

Outline of proof

Since $J(v)$ is differentiable and U_{ad} is bounded, then the optimal control u are characterized by $J'(u)(v - u) \geq 0 \quad \forall v \in U_{ad}$ which is equivalent to $(Nu, v - u)_{[L^2(\Omega)]^2} +$

$$(y_1(u) - z_{1d}, y_1(v) - y_1(u)) \geq 0 \qquad (8)$$

Since $(A^*p, y) = (p, Ay)$ where A is defined by:

$A\varphi = A(y_1, y_2) = (-\Delta y_1 + ay_1 - by_2, -\Delta y_2 + by_1 + ay_2)$

, then

$(p, Ay) =$

$$((p_1, -\Delta y_1 + ay_1 - by_2), (p_2, -\Delta y_2 + by_1 + ay_2))$$

By Green's formula or derivative in the sense of distribution

$$A\varphi = (-\Delta p_1 + ap_1 - bp_2, y_1) + (-\Delta p_2 + bp_1 + ap_2, y_2)$$

= (A^*p, y), where

$A^*p = A^*\{p_1(u), p_2(u)\} = y(u) - z_d$ then

$Ap(u) + M^T p(u) = Y(u) - Z_d$,

$A = -\Delta, M = a_{ij}, i, j = 1,2$, (8) is equivalent to

$$(-\Delta p_1 + ap_1 - bp_2, y_1(v) - y_1(u)) + (-\Delta p_2 + bp_1 + ap_2, y_2(v) - y_2(u)) + (Nu, v - u)_{[L^2(\Omega)]^2} \geq 0.$$

$$(p_1, -\Delta y_1(v) + \Delta y_2(u)) + (p_2, -\Delta y_2(v) + \Delta y_2(u)) + (ap_1 + bp_2, y_2(v) - y_2(u)) + (bp_1 - ap_2, y_1(v) - y_1(u)) + (Nu, v - u)_{[L^2(\Omega)]^2} \geq 0$$

From (7), we obtain

$$\int_\Omega \{p_1(v_1 - u_1) + p_2(v_2 - u_2)\} dx + (Nu, v - u)_{[L^2(\Omega)]^2} \geq 0.$$

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Remark 1:

1. Generalization to $n \times n$ systems

AY +MY=F in Ω
Y=0, $\frac{\partial Y}{\partial \nu_A} = g$ on Γ
and conjugation conditions

$\left[\frac{\partial Y}{\partial \nu_A}\right]=\left[\sum_{i,k=1}^{n}\frac{\partial y_i}{\partial x_k}\cos(\nu,x_k)\right]=0$
$\left\{\sum_{i,k=1}^{n}\frac{\partial y_i}{\partial x_k}\cos(\nu,x_k)\right\}^{\pm}=r[y]$
,

where Ω is an open subset of R^n with smooth boundary Γ, A is an n×n diagonal matrix of Laplace operator$(-\Delta)$, M=$(a_{ij})_{i,j=1}^{n}$ is an $n \times n$ matrix with coefficients defined by:-

$$a_{ij} = \begin{cases} a & i=j \\ -b_i & i<j \\ b_i & i>j \end{cases} \#$$

$\Gamma = (\partial\Omega_1 \cup \partial\Omega_2)\backslash\gamma$, $\gamma = \partial\Omega_1 \cup \partial\Omega_2 \neq \phi, \cos(\nu,x_k)$ =K−the direction cosine of ν, ν being the normal on the Γ exterior to Ω, $[\varphi] = \varphi^+ - \varphi^-, \{\varphi(z)\}^+ = \varphi(z+0)$ and

$\{\varphi(z)\}^- = \varphi(z-0)$. In this case, the bilinear form is given by:
$a(y,\Psi)=$

$$\sum_{i=1}^{n}\int_{\Omega}(-\Delta y_i)\Psi_i\,dx + \int_{\gamma} r[y_i][\Psi_i]\,d\gamma + \sum_{i,j=1}^{n}\int_{\Gamma} F(y_i,\Psi_i)\,d\Gamma + \sum_{i,j=1}^{n}\int_{\Omega} a_{ij}\, y_j\Psi_i\,dx,$$

The linear form is given by:

$$L(\Psi) = \sum_{i=1}^{n}\int_{\Omega}(f_i + g_i)\Psi_i\,dx.$$

The cost functional is given by: J (u) $=(\bar{a}u,u)_{[L^2(\Omega)]^n} + \|y(u) - z_d\|_H^2$

J (u) $=\sum_{i=1}^{n}\int_{\Omega}(y_i(u) - z_{id})^2\,dx + (\bar{a}u,u)_{[L^2(\Omega)]^n}$,

, where $Z_d = \{Z_{1d}, Z_{2d}, \ldots\ldots, Z_{nd}\}$ in $[L^2(\Gamma)]^n$, H$\subset L_2(\Omega)$, $\bar{a}$ is hermitian positive definite operator defined on $[L^2(\Omega)]^2$ such that:
$(\bar{a}u,u)_{[L^2(\Omega)]^2} \geq \gamma\|u\|^2_{[L^2(\Omega)]^2}$,$\gamma > 0$.

In this case the necessary and sufficient for u to be an optimal control is that the following equations and inequalities be satisfied.

$$\begin{cases} Ap(u) + M^T p(u) = Y(u) - Z_d & \text{in } \Omega \\ p(u) = 0, \frac{\partial p_i}{\partial \nu_{A^*}} = 0 & \text{on } \Gamma \\ \left[\sum_{i,k=1}^{n}\frac{\partial p_i}{\partial x_k}\cos(\nu,x_k)\right] = 0, \left\{\sum_{i,k=1}^{n}\frac{\partial p_i}{\partial x_k}\cos(\nu,x_k)\right\}^{\pm} = r[p] \end{cases}$$

M=$(a_{ij})_{i,j=1}^{n}$, M^T= transpose of M.

2. If constraints are absent i.e. when $u_g = u$ the equality $p(u) + \bar{a}u = 0$ then$-\frac{p(u)}{a} = u$ then the differential problem of finding the vector-function $\#(y,p)^T$, that satisfies the relations

$$\begin{cases} Ay(u) + \frac{p(u)}{a} + My(u) = F & \text{in } \Omega \\ Ap(u) + M^T p(u) - y(u) = -z_d & \text{in } \Omega \\ p(u) = y(u) = 0, \frac{\partial y}{\partial \nu_A} = g, \frac{\partial p}{\partial \nu_{A^*}} = 0 & \text{on } \Gamma \\ \left[\sum_{i,k=1}^{n}\frac{\partial y_i}{\partial x_k}\cos(\nu,x_k)\right] = 0, \left\{\sum_{i,k=1}^{n}\frac{\partial y_i}{\partial x_k}\cos(\nu,x_k)\right\}^{\pm} = r[y], x \in \gamma \\ \left\{\sum_{i,k=1}^{n}\frac{\partial p_i}{\partial x_k}\cos(\nu,x_k)\right\}^{\pm} = 0, \left\{\sum_{i,k=1}^{n}\frac{\partial p_i}{\partial x_k}\cos(\nu,x_k)\right\}^{\pm} = r[p], x \in \gamma \end{cases} \quad (9)$$

3. we can find last relations at n= 2 as the same in the first paper.

We can choose a=b=1 (special case) and with Neumann conditions can get distributed control.

Conclusion

Finding the existence and uniqueness for solution (state of the system) and necessary and sufficient condition for optimality of the system in 2×2 and generalization to n×n

The following work will be offer the boundary control in non cooperative system.

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