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On the Inverse Tangent Function, π and Babylonian Identity

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ABSTRACT

We evaluate the constant π using the Babylonian identity and the inverse cosine or sine function.

Keywords: *Inverse Tangent Function, Babylonian Identity*

1. INTRODUCTION

By means of the inverse cosine function and Babylonian identity, we demonstrated the identities following, among others:

$$\frac{\pi}{2}=\sin^{-1}\left(\frac{1}{3}\right)+\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right),$$

$$\frac{\pi}{2}=\sin^{-1}\left(\frac{1}{4}\right)+\sin^{-1}\left(\frac{\sqrt{15}}{4}\right),$$

$$\frac{\pi}{2}=\sin^{-1}\left(\frac{1}{5}\right)+\sin^{-1}\left(\frac{2\sqrt{6}}{5}\right),$$

$$\frac{\pi}{2}=\sin^{-1}\left(\frac{1}{6}\right)+\sin^{-1}\left(\frac{\sqrt{35}}{6}\right),$$

$$\frac{\pi}{2}=\sin^{-1}\left(\frac{1}{7}\right)+\sin^{-1}\left(\frac{4\sqrt{3}}{7}\right),$$

$$\frac{\pi}{2}=\sin^{-1}\left(\frac{1}{8}\right)+\sin^{-1}\left(\frac{3\sqrt{7}}{8}\right),$$

$$\frac{\pi}{2}=\sin^{-1}\left(\frac{1}{9}\right)+\sin^{-1}\left(\frac{4\sqrt{5}}{9}\right),$$

$$\frac{\pi}{2}=\sin^{-1}\left(\frac{1}{10}\right)+\sin^{-1}\left(\frac{3\sqrt{11}}{10}\right),$$

and so on.

2. THEOREMS

Theorem 1: For $0\le \operatorname{Re}(x)\le 1$ and $\operatorname{Im}(x)=0$, then

$$\cos^{-1}(x)=\frac{\pi}{2}-\sum_{n=0}^{\infty}\frac{(2n)!}{2^{2n}(n!)^2(2n+1)}x^{2n+1},$$

Where $\cos^{-1}(x)$ denotes the inverse cosine function.

Proof: In previous paper [1], we put $\frac{a-b}{a+b}=t$ in (5), and encounter

$$\frac{1}{\sqrt{1-t^2}}=\sum_{n=0}^{\infty}\frac{(2n)!}{2^{2n}n!^2}t^{2n}. \quad (8)$$

We integrate (8) from x at 1 in t, then

$$\int_x^1\frac{dt}{\sqrt{1-t^2}}=\sum_{n=0}^{\infty}\frac{(2n)!}{2^{2n}n!^2}\int_x^1 t^{2n}\,dt. \quad (9)$$

We calculate

$$\int_x^1\frac{dt}{\sqrt{1-t^2}}=\cos^{-1}(x) \quad (10)$$

And

$$\int_x^1 t^{2n}\,dt=\frac{1-x^{2n+1}}{2n+1}. \quad (11)$$

From (9), (10) and (11), it follows

$$\cos^{-1}(x)=\sum_{n=0}^{\infty}\frac{(2n)!}{2^{2n}(n!)^2(2n+1)}-\sum_{n=0}^{\infty}\frac{(2n)!}{2^{2n}(n!)^2(2n+1)}x^{2n+1}$$

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$$= \frac{\pi}{2} - \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n}(n!)^2(2n+1)} x^{2n+1}. \quad (12)$$

Theorem 2: For $x \in [0,1]$, then

$$\cos^{-1}(x) = \frac{\pi}{2} - x\sqrt{1-x^2} - 4x \sum_{n=0}^{\infty} \frac{(2n)!\, n}{2^{2n}(n!)^2(4n^2-1)} x^{2n},$$

Where $\sin^{-1}(x)$ denotes the inverse sine function.

Proof: Using the identity

$$\frac{1}{2n+1} = \frac{4n}{4n^2-1} - \frac{1}{2n-1} \quad (13)$$

in the right-hand side of Theorem 1, we have

$$\cos^{-1}(x) = \frac{\pi}{2} - \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n}(n!)^2(2n+1)} x^{2n+1}$$
$$= \frac{\pi}{2} - 4x \sum_{n=0}^{\infty} \frac{(2n)!\, n}{2^{2n}(n!)^2(4n^2-1)} x^{2n} + x \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n}(n!)^2(2n-1)} x^{2n}.$$

We calculate

$$\sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n}(n!)^2(2n-1)} x^{2n} = -\sqrt{1-x^2}. \quad (14)$$

We substitute (14) into (13), and obtain

$$\cos^{-1}(x) = \frac{\pi}{2} - x\sqrt{1-x^2} - 4x \sum_{n=0}^{\infty} \frac{(2n)!\, n}{2^{2n}(n!)^2(4n^2-1)} x^{2n}.$$

Theorem 3: .For $x \in [0,\pi]$, then

$$\cos^{-1}(\sin x) = \frac{\pi - \sin(2x)}{2} - 4\sin x \sum_{n=1}^{\infty} \frac{(2n)!\, n}{2^{2n}(n!)^2(4n^2-1)} (\sin x)^{2n},$$

Where $\cos^{-1}(x)$ denotes the inverse sine function, $\sin(x)$ denotes the sine function and $\cos(x)$ denotes the cosine function.

Proof: Let $x = \sin x$ in Theorem 2

$$4\sum_{n=1}^{\infty} \frac{(2n)!\, n}{2^{2n}(n!)^2(4n^2-1)} (\sin x)^{2n+1} = \frac{\pi}{2} - \frac{\sin(2x)}{2} - \cos^{-1}(\sin x) \Rightarrow$$

$$\cos^{-1}(\sin x) = \frac{\pi - \sin(2x)}{2} - 4\sin x \sum_{n=1}^{\infty} \frac{(2n)!\, n}{2^{2n}(n!)^2(4n^2-1)} (\sin x)^{2n}. \quad (15)$$

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Explicit Calculations.we leave for the reader:

$$\sum_{n=1}^{\infty}\frac{(2n)!\,n}{2^{2n}(n!)^2(4n^2-1)}\left(\frac{\sqrt{2+\sqrt{2}}}{2}\right)^{2n}=\frac{3\pi-2\sqrt{2}}{16\sqrt{2+\sqrt{2}}},$$

$$\sum_{n=1}^{\infty}\frac{(2n)!\,n}{2^{2n}(n!)^2(4n^2-1)}\left(\sqrt{\frac{5}{8}+\frac{\sqrt{5}}{8}}\right)^{2n}=\frac{8\sqrt{2}\pi-5\sqrt{5-\sqrt{5}}}{40\sqrt{5+\sqrt{5}}},$$

$$\sum_{n=1}^{\infty}\frac{(2n)!\,n}{2^{2n}(n!)^2(4n^2-1)}\left(\frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2}\right)^{2n}=\frac{7\pi-4\sqrt{2-\sqrt{2}}}{32\sqrt{2+\sqrt{2+\sqrt{2}}}},$$

$$\sum_{n=1}^{\infty}\frac{(2n)!\,n}{2^{2n}(n!)^2(4n^2-1)}\left(\frac{1+\sqrt{3}}{2\sqrt{2}}\right)^{2n}=\frac{\sqrt{2}(5\pi-3)}{24(1+\sqrt{3})},$$

$$\sum_{k=1}^{\infty}\frac{(2n)!\,n}{2^{2n}(n!)^2(4n^2-1)}\left(\frac{1+\sqrt{5}}{4\sqrt{2}}+\frac{\sqrt{5-\sqrt{5}}}{4}\right)^{2n}=\frac{18\sqrt{2}\pi-5\left(\sqrt{10}-\sqrt{2}\right)}{40\left(1+\sqrt{5}+\sqrt{10-2\sqrt{5}}\right)},$$

$$\sum_{n=1}^{\infty}\frac{(2n)!\,n}{2^{2n}(n!)^2(4n^2-1)}\left(\frac{\sqrt{5+\sqrt{5}}}{4}-\frac{1-\sqrt{5}}{4\sqrt{2}}\right)^{2n}=\frac{14\pi-5\left(1+\sqrt{5}\right)}{20\left(2\sqrt{5+\sqrt{5}}+\sqrt{10}-\sqrt{2}\right)}$$

And

$$\sum_{n=1}^{\infty}\frac{(2n)!\,n}{2^{2n}(n!)^2(4n^2-1)}\left(\frac{\sqrt{3}}{2}\right)^{2n}=\frac{4\pi-3\sqrt{3}}{24\sqrt{3}},$$

Consequently,

$$\pi=\frac{2\sqrt{2}}{3}+\frac{16\sqrt{2+\sqrt{2}}}{3}\sum_{n=1}^{\infty}\frac{(2n)!\,n}{2^{2n}(n!)^2(4n^2-1)}\left(\frac{\sqrt{2+\sqrt{2}}}{2}\right)^{2n},$$

$$\pi=\frac{5}{8}\sqrt{\frac{5}{2}-\frac{\sqrt{5}}{2}}+5\sqrt{\frac{5}{2}+\frac{\sqrt{5}}{2}}\sum_{n=1}^{\infty}\frac{(2n)!\,n}{2^{2n}(n!)^2(4n^2-1)}\left(\sqrt{\frac{5}{8}+\frac{\sqrt{5}}{8}}\right)^{2n},$$

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$$\pi = \frac{4}{7}\sqrt{2-\sqrt{2}} + \frac{32}{7}\sqrt{2+\sqrt{2+\sqrt{2}}}\sum_{n=1}^{\infty}\frac{(2n)!\,n}{2^{2n}(n!)^2(4n^2-1)}\left(\frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2}\right)^{2n},$$

$$\pi = \frac{3}{5} + \frac{24\left(1+\sqrt{3}\right)}{5\sqrt{2}}\sum_{n=1}^{\infty}\frac{(2n)!\,n}{2^{2n}(n!)^2(4n^2-1)}\left(\frac{1+\sqrt{3}}{2\sqrt{2}}\right)^{2n},$$

$$\pi = \frac{5\left(\sqrt{5}-1\right)}{18} + \frac{20\left(1+\sqrt{5}+\sqrt{10-2\sqrt{5}}\right)}{9\sqrt{2}}\sum_{n=1}^{\infty}\frac{(2n)!\,n}{2^{2n}(n!)^2(4n^2-1)}\left(\frac{1+\sqrt{5}}{4\sqrt{2}}+\frac{\sqrt{5-\sqrt{5}}}{4}\right)^{2n},$$

$$\pi = \frac{5\left(1+\sqrt{5}\right)}{14} + \frac{10}{7}\left(2\sqrt{5+\sqrt{5}}+\sqrt{10}-\sqrt{2}\right)\sum_{k=1}^{\infty}\frac{(2n)!\,n}{2^{2n}(n!)^2(4n^2-1)}\left(\frac{\sqrt{5+\sqrt{5}}}{4}-\frac{1-\sqrt{5}}{4\sqrt{2}}\right)^{2n}$$

And

$$\pi = \frac{3\sqrt{3}}{4} + 6\sqrt{3}\sum_{n=1}^{\infty}\frac{(2n)!\,n}{2^{2n}(n!)^2(4n^2-1)}\left(\frac{\sqrt{3}}{2}\right)^{2n},$$

just as in the previous paper [1].

Corollary 1: we have

$$\pi^2 = 4 + 16\sqrt{\pi}\sum_{n=1}^{\infty}\frac{(2n)!\,n}{2^{2n}n!\,(4n^2-1)\Gamma\left(n+\frac{3}{2}\right)} = 4 + 32\sum_{n=1}^{\infty}\frac{n}{(2n-1)(2n+1)^2},$$

Proof: we integrate the Theorem 3 from 0 at $\frac{\pi}{2}$ with respect to x

$$\int_0^{\frac{\pi}{2}}\cos^{-1}(\sin x)\,dx = \int_0^{\frac{\pi}{2}}\frac{\pi-\sin(2x)}{2}\,dx - 4\sum_{n=1}^{\infty}\frac{(2n)!\,n}{2^{2n}(n!)^2(4n^2-1)}\int_0^{\frac{\pi}{2}}(\sin x)^{2n+1}\,dx. \quad (16)$$

We calculate

$$\int_0^{\frac{\pi}{2}}\cos^{-1}(\sin x)\,dx = \frac{\pi^2}{8}, \quad (17)$$

$$\int_0^{\frac{\pi}{2}}\frac{\pi-\sin(2x)}{2}\,dx = \frac{\pi^2-2}{4}, \quad (18)$$

And

$$\int_0^{\frac{\pi}{2}}(\sin x)^{2k+1}\,dx = \frac{\sqrt{\pi}\,\Gamma(k+1)}{2\Gamma\left(k+\frac{3}{2}\right)}, \qquad (19)$$

for $k > -1$.

From (16), (17), (18) and (19), the proof follows.

3. NOTE ON PREVIOUS PAPER

In [1], wemay expand the summation and encounter the explicit evaluation of the inverse sine function:

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$$\sin^{-1}\left(\frac{\sqrt{2+\sqrt{3}}}{2}\right)=\frac{3\pi+4\sqrt{3-2\sqrt{2}}+2\sqrt{6-4\sqrt{2}}-3\sqrt{2}}{8},$$

$$\sin^{-1}\left(\sqrt{\frac{5+\sqrt{5}}{8}}\right)=\frac{2\pi}{5}+\frac{\sqrt{5-2\sqrt{5}}+\sqrt{25-10\sqrt{5}}}{8}-\frac{1}{4}\sqrt{\frac{10}{5+\sqrt{5}}}$$

$$\sin^{-1}\left(\frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2}\right)=\frac{7\pi}{16}+\frac{1}{4}\sqrt{\frac{(2+\sqrt{2})(2-\sqrt{2+\sqrt{2}})}{2+\sqrt{2+\sqrt{2}}}}+\frac{1}{2}\sqrt{\frac{2-\sqrt{2+\sqrt{2}}}{2+\sqrt{2+\sqrt{2}}}}-\frac{\sqrt{2-\sqrt{2}}}{4},$$

$$\sin^{-1}\left(\frac{1+\sqrt{3}}{2\sqrt{2}}\right)=\frac{5\pi-3}{12}+\frac{1+\sqrt{3}}{4}\sqrt{\frac{2-\sqrt{3}}{2}},$$

$$\sin^{-1}\left(\frac{2\sqrt{5-\sqrt{5}}+\sqrt{2}+\sqrt{10}}{8}\right)=\frac{9\pi}{20}+\frac{\sqrt{16-2\sqrt{10-2\sqrt{5}}-2\sqrt{50-10\sqrt{5}}}}{32}$$

$$+\frac{\sqrt{80-10\sqrt{10-2\sqrt{5}}-10\sqrt{50-10\sqrt{5}}}}{32}+\frac{\sqrt{10-2\sqrt{5}-\sqrt{50-10\sqrt{5}}}}{8}$$

$$-\frac{4+\sqrt{50-10\sqrt{5}}-\sqrt{10-2\sqrt{5}}}{8\left(\sqrt{10-2\sqrt{5}}+\sqrt{5}+1\right)},$$

$$\sin^{-1}\left(\frac{2\sqrt{5+\sqrt{5}}-\sqrt{2}+\sqrt{10}}{8}\right)=$$

$$\frac{7\pi}{20}+\frac{\sqrt{10+2\sqrt{5}-\sqrt{50+10\sqrt{5}}}}{8}+\frac{5\sqrt{80+10\sqrt{10+2\sqrt{5}}-10\sqrt{50+10\sqrt{5}}}}{160}$$

$$-\frac{5\sqrt{16+2\sqrt{10+2\sqrt{5}}-2\sqrt{50+10\sqrt{5}}}}{160}-\frac{\sqrt{5}}{8}-\frac{1}{8}.$$

Theorem 3.1: For $x\in[0,1]$, then

$$\cos^{-1}(x)=x\sqrt{1-x^2}+4\sqrt{1-x^2}\sum_{n=1}^{\infty}\frac{(2n)!\,n}{2^{2n}(n!)^2(4n^2-1)}(1-x^2)^n$$

Or

$$\sum_{n=1}^{\infty}\frac{(2n)!\,n}{2^{2n}(n!)^2(4n^2-1)}(1-x^2)^n=\frac{\cos^{-1}(x)-x\sqrt{1-x^2}}{4\sqrt{1-x^2}},$$

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Where $\cos^{-1}(x)$ denotes the inverse cosine function.

Proof: In previous paper [1], we proved that

$$\sin^{-1}(x)=x\sqrt{1-x^2}+4x\sum_{n=1}^{\infty}\frac{(2n)!\,n}{2^{2n}(n!)^2(4n^2-1)}x^{2n}. \quad (3.1)$$

We set $\sin x$ by x into (3.1)

$$x=\frac{\sin(2x)}{2}+4\sin x\sum_{n=1}^{\infty}\frac{(2n)!\,n}{2^{2n}(n!)^2(4n^2-1)}(\sin x)^{2n}. \quad (3.2)$$

We replace $\cos^{-1}x$ by x into (3.2)

$$\cos^{-1}x=\frac{\sin(2\cos^{-1}x)}{2}+4\sin(\cos^{-1}x)\sum_{n=1}^{\infty}\frac{(2n)!\,n}{2^{2n}(n!)^2(4n^2-1)}[\sin(\cos^{-1}x)]^{2n}$$

$$=x\sqrt{1-x^2}+4\sqrt{1-x^2}\sum_{n=1}^{\infty}\frac{(2n)!\,n}{2^{2n}(n!)^2(4n^2-1)}\left(\sqrt{1-x^2}\right)^{2n}$$

$$=x\sqrt{1-x^2}+4\sqrt{1-x^2}\sum_{n=1}^{\infty}\frac{(2n)!\,n}{2^{2n}(n!)^2(4n^2-1)}(1-x^2)^{n}. \quad (3.3)$$

Explicit Evaluation: Let $x=\frac{1+\sqrt{3}}{4}$ in Theorem 3.1

$$\sum_{n=1}^{\infty}\frac{(2n)!\,n}{2^{2n}(n!)^2(4n^2-1)}\left(\frac{3}{4}\right)^n=\frac{4\pi-3\sqrt{3}}{24\sqrt{3}},$$

and so on. we may use the Theorem 3.1 and expansion of the summation to obtain the identities

$$\cos^{-1}\left(\frac{1}{3}\right)=\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right),$$

$$\cos^{-1}\left(\frac{1}{4}\right)=\sin^{-1}\left(\frac{\sqrt{15}}{4}\right),$$

$$\cos^{-1}\left(\frac{1}{5}\right)=\sin^{-1}\left(\frac{2\sqrt{6}}{5}\right),$$

$$\cos^{-1}\left(\frac{1}{6}\right)=\sin^{-1}\left(\frac{\sqrt{35}}{6}\right),$$

$$\cos^{-1}\left(\frac{1}{7}\right)=\sin^{-1}\left(\frac{4\sqrt{3}}{7}\right),$$

$$\cos^{-1}\left(\frac{1}{8}\right)=\sin^{-1}\left(\frac{3\sqrt{7}}{8}\right),$$

$$\cos^{-1}\left(\frac{1}{9}\right)=\sin^{-1}\left(\frac{4\sqrt{5}}{9}\right),$$

$$\cos^{-1}\left(\frac{1}{10}\right)=\sin^{-1}\left(\frac{3\sqrt{11}}{10}\right),$$

and so on. Using the identity $\sin^{-1}x+\cos^{-1}x=\frac{\pi}{2}$ in previous identities, we have

$$\frac{\pi}{2}=\sin^{-1}\left(\frac{1}{3}\right)+\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right),$$

$$\frac{\pi}{2}=\sin^{-1}\left(\frac{1}{4}\right)+\sin^{-1}\left(\frac{\sqrt{15}}{4}\right),$$

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$$\frac{\pi}{2} = \sin^{-1}\left(\frac{1}{5}\right) + \sin^{-1}\left(\frac{2\sqrt{6}}{5}\right),$$

$$\frac{\pi}{2} = \sin^{-1}\left(\frac{1}{6}\right) + \sin^{-1}\left(\frac{\sqrt{35}}{6}\right),$$

$$\frac{\pi}{2} = \sin^{-1}\left(\frac{1}{7}\right) + \sin^{-1}\left(\frac{4\sqrt{3}}{7}\right),$$

$$\frac{\pi}{2} = \sin^{-1}\left(\frac{1}{8}\right) + \sin^{-1}\left(\frac{3\sqrt{7}}{8}\right),$$

$$\frac{\pi}{2} = \sin^{-1}\left(\frac{1}{9}\right) + \sin^{-1}\left(\frac{4\sqrt{5}}{9}\right),$$

$$\frac{\pi}{2} = \sin^{-1}\left(\frac{1}{10}\right) + \sin^{-1}\left(\frac{3\sqrt{11}}{10}\right),$$

And so on; as well as

$$\frac{\pi}{2} = \cos^{-1}\left(\frac{1}{3}\right) + \cos^{-1}\left(\frac{2\sqrt{2}}{3}\right),$$

$$\frac{\pi}{2} = \cos^{-1}\left(\frac{1}{4}\right) + \cos^{-1}\left(\frac{\sqrt{15}}{4}\right),$$

$$\frac{\pi}{2} = \cos^{-1}\left(\frac{1}{5}\right) + \cos^{-1}\left(\frac{2\sqrt{6}}{5}\right),$$

$$\frac{\pi}{2} = \cos^{-1}\left(\frac{1}{6}\right) + \cos^{-1}\left(\frac{\sqrt{35}}{6}\right),$$

$$\frac{\pi}{2} = \cos^{-1}\left(\frac{1}{7}\right) + \cos^{-1}\left(\frac{4\sqrt{3}}{7}\right),$$

$$\frac{\pi}{2} = \cos^{-1}\left(\frac{1}{8}\right) + \cos^{-1}\left(\frac{3\sqrt{7}}{8}\right),$$

$$\frac{\pi}{2} = \cos^{-1}\left(\frac{1}{9}\right) + \cos^{-1}\left(\frac{4\sqrt{5}}{9}\right),$$

$$\frac{\pi}{2} = \cos^{-1}\left(\frac{1}{10}\right) + \cos^{-1}\left(\frac{3\sqrt{11}}{10}\right),$$

and so on.

Theorem 3.2.For $x \in [0,1]$, then

$$\sin^{-1}(x) = x\sqrt{1-x^2} + 2x^3\sum_{n=0}^{\infty}\frac{(2n+2)!}{2^{2n+1}(2n+3)(2n+1)(n+1)}\frac{x^{2n}}{(n!)^2}$$

Or

$$\sum_{n=0}^{\infty}\frac{(2n+2)!}{2^{2n+1}(2n+3)(2n+1)(n+1)}\frac{x^{2n}}{(n!)^2} = \frac{\sin^{-1}(x) - x\sqrt{1-x^2}}{2x^3},$$

Where $\sin^{-1}(x)$ denotes the inverse sine function.

Proof: In previous paper [1], we proved that

$$\sin^{-1}(x) = x\sqrt{1-x^2} + 4x\sum_{n=1}^{\infty}\frac{(2n)!\,n}{2^{2n}(n!)^2(4n^2-1)}x^{2n}, \quad (3.4)$$

Proof: In previous paper [1], we proved that

$$\sin^{-1}(x) = x\sqrt{1-x^2} + 4x\sum_{n=1}^{\infty}\frac{(2n)!\,n}{2^{2n}(n!)^2(4n^2-1)}x^{2n}, \quad (3.5)$$

We calculate the identity

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$$\frac{1}{4n^2-1}=\int_0^1 \frac{1}{(2n+2v-1)^2}dv. \quad (3.6)$$

We set (3.6) into (3.5)

$$\sin^{-1}(x)=x\sqrt{1-x^2}+4x\sum_{n=1}^{\infty}\frac{(2n)!\,n}{2^{2n}(n!)^2}\int_0^1\frac{1}{(2n+2v-1)^2}dv\,x^{2n}$$

$$=x\sqrt{1-x^2}+4x\int_0^1\sum_{n=1}^{\infty}\frac{(2n)!\,n}{2^{2n}(n!)^2(2n+2v-1)^2}x^{2n}\,dv$$

$$=x\sqrt{1-x^2}+2x^3\int_0^1\frac{1}{(2v+1)^2}\,{}_3F_2\left(\begin{matrix}\frac{3}{2},v+\frac{1}{2},v+\frac{1}{2}\\ v+\frac{3}{2},v+\frac{3}{2}\end{matrix}\middle|x^2\right)dv$$

$$=x\sqrt{1-x^2}+2x^3\int_0^1\frac{1}{(2v+1)^2}\sum_{n=0}^{\infty}\frac{\left(\frac{3}{2}\right)_n\left(v+\frac{1}{2}\right)_n\left(v+\frac{1}{2}\right)_n}{\left(v+\frac{3}{2}\right)_n\left(v+\frac{3}{2}\right)_n}\frac{x^{2n}}{n!}dv$$

$$=x\sqrt{1-x^2}+2x^3\sum_{n=0}^{\infty}\left(\frac{3}{2}\right)_n\int_0^1\frac{\left(v+\frac{1}{2}\right)_n\left(v+\frac{1}{2}\right)_n}{(2v+1)^2\left(v+\frac{3}{2}\right)_n\left(v+\frac{3}{2}\right)_n}dv\,\frac{x^{2n}}{n!}$$

$$=x\sqrt{1-x^2}+2x^3\sum_{n=0}^{\infty}\frac{\left(\frac{3}{2}\right)_n}{4n^2+8n+3}\frac{x^{2n}}{n!}$$

$$=x\sqrt{1-x^2}+2x^3\sum_{n=0}^{\infty}\frac{1}{4n^2+8n+3}\frac{(2n+2)!}{2^{2n+1}(n+1)!}\frac{x^{2n}}{n!}$$

$$=x\sqrt{1-x^2}+2x^3\sum_{n=0}^{\infty}\frac{(2n+2)!}{2^{2n+1}(2n+3)(2n+1)(n+1)}\frac{x^{2n}}{(n!)^2}. \quad (3.7)$$

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