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Other Integral Representation for the Digamma Function and Special Values

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ABSTRACT

We proved two news integral representations for the digamma function using a method non-orthodox of substitution.

Keywords: *Integral, digamma function*

1. INTRODUCTION

Using a method non-orthodox of substitution, we proved the following integral representations of the digamma function:

$$\psi(s) = \int_0^\infty \left[\frac{1}{(u+1)^2 \ln(u+1)} - \frac{(u+1)^{-s}}{u}\right] du$$

And

$$\psi(s) = \int_0^\infty \left[\frac{1}{e^t t} - \frac{e^{t(1-s)}}{e^t - 1}\right] du,$$

as well as some unknown special values in the current mathematical literature:

$$\psi\left(\frac{3}{2}\right) = 2 - \gamma - 2\ln 2,$$

$$\psi\left(\frac{4}{3}\right) = 3 - \frac{\pi}{2\sqrt{3}} - \gamma - \frac{3\ln 3}{2},$$

$$\psi\left(\frac{5}{4}\right) = 4 - \frac{\pi}{2} - \gamma - 3\ln 2,$$

$$\psi\left(\frac{7}{6}\right) = 6 - \frac{\sqrt{3}\,\pi}{2} - \gamma - 2\ln 2 - \frac{3\ln 3}{2},$$

$$\psi\left(\frac{9}{8}\right) = 8 - \frac{(1+\sqrt{2})\,\pi}{2} - \gamma - 4\ln 2 - \sqrt{2}\coth^{-1}(\sqrt{2}),$$

$$\psi\left(\frac{11}{10}\right) = 10 - \frac{\sqrt{5+2\sqrt{5}}\,\pi}{2} - \gamma - 2\ln 2 - \frac{5\ln 5}{4} - \frac{\sqrt{5}}{2}\tanh^{-1}\left(\frac{2}{\sqrt{5}}\right),$$

inter alia.

2. THEOREM

Theorem 1: For $\Re(s) > 0$, then

$$\psi(s) = \int_0^\infty \left[\frac{1}{(u+1)^2 \ln(u+1)} - \frac{(u+1)^{-s}}{u}\right] du,$$

where $\psi(s)$ denotes the digamma function and $\ln s$ denotes the natural logarithm.

Proof: In [1], the digamma function is defined by

$$\psi(s) \stackrel{\text{def}}{=} \int_0^1 \left(\frac{t^{s-1}}{t-1} - \frac{1}{\ln t}\right) dt. \quad (1)$$

Take $t = \frac{a+bu}{1+u}$ and $dt = \left[\frac{b}{1+u} - \frac{a+bu}{(1+u)^2}\right] du$ in (1);thus, I obtain

$$\psi(s) = \int_{-\frac{a}{b}}^{-\frac{a-1}{b-1}} \left[\frac{\left(\frac{a+bu}{1+u}\right)^{s-1}}{\frac{a+bu}{1+u} - 1} - \frac{1}{\ln\left(\frac{a+bu}{1+u}\right)}\right]\left[\frac{b}{1+u} - \frac{a+bu}{(1+u)^2}\right] du. \quad (2)$$

Let $-\frac{a-1}{b-1} = \tan(\pi\theta)$ and $-\frac{a}{b} = \cot(\pi\theta)$, therefore,

$$a = \frac{(\tan(\pi\theta)+1)\cot(\pi\theta)}{\cot(\pi\theta) - \tan(\pi\theta)}, \quad b = -\frac{\tan(\pi\theta)+1}{\cot(\pi\theta) - \tan(\pi\theta)}. \quad (3)$$

Put (3) in (2) and simplifying, we have

$$\psi(s) = \int_{\cot(\pi\theta)}^{\tan(\pi\theta)} f(u,\theta)\,du, \quad (4)$$

Where

$$f(u,\theta) = -\frac{2(\csc(2\pi\theta)+1)\left(-\frac{2(u+1)^2(\sin(2\pi\theta)-1)\left(\frac{\cos(\pi\theta)-u\sin(\pi\theta)}{(u+1)(\cos(\pi\theta)-\sin(\pi\theta))}\right)^s}{(u^2+1)\sin(2\pi\theta)-2u} - \frac{1}{\ln\left(\frac{\cos(\pi\theta)-u\sin(\pi\theta)}{(u+1)(\cos(\pi\theta)-\sin(\pi\theta))}\right)}\right)}{(u+1)^2(\cot(\pi\theta)-\tan(\pi\theta))}.$$

Consider the limit $\theta \to 1$ in (4)

$$\lim_{\theta\to 1}\psi(s) = \lim_{\theta\to 1}\int_{\cot(\pi\theta)}^{\tan(\pi\theta)} f(u,\theta)\,du, \quad (5)$$

Then

$$\lim_{\theta\to 1}\psi(s) = \psi(s), \quad (6)$$

$$\lim_{\theta\to 1} f(u,\theta) = -\frac{-\frac{\left(\frac{1}{u+1}\right)^{s-2}}{u} - \frac{1}{\ln\left(\frac{1}{u+1}\right)}}{(u+1)^2},$$

$$\lim_{\theta\to 1}\tan(\pi\theta) = 0,$$

$$\lim_{\theta\to 1}\cot(\pi\theta) = \infty.$$

Hence, from (5) and (6), it follows that

$$\psi(s) = \int_0^\infty \left[\frac{1}{(u+1)^2\ln(u+1)} - \frac{(u+1)^{-s}}{u}\right] du.$$

Special Values: Using the Theorem 1, we calculate that

$$\psi\left(\frac{3}{2}\right) = 2 - \gamma - 2\ln 2,$$

$$\psi\left(\frac{4}{3}\right) = 3 - \frac{\pi}{2\sqrt{3}} - \gamma - \frac{3\ln 3}{2},$$

$$\psi\left(\frac{5}{4}\right) = 4 - \frac{\pi}{2} - \gamma - 3\ln 2,$$

$$\psi\left(\frac{7}{6}\right) = 6 - \frac{\sqrt{3}\,\pi}{2} - \gamma - 2\ln 2 - \frac{3\ln 3}{2},$$

$$\psi\left(\frac{9}{8}\right) = 8 - \frac{(1+\sqrt{2})\,\pi}{2} - \gamma - 4\ln 2 - \sqrt{2}\coth^{-1}(\sqrt{2}),$$

$$\psi\left(\frac{11}{10}\right) = 10 - \frac{\sqrt{5+2\sqrt{5}}\,\pi}{2} - \gamma - 2\ln 2 - \frac{5\ln 5}{4} - \frac{\sqrt{5}}{2}\tanh^{-1}\left(\frac{2}{\sqrt{5}}\right).$$

Corollary 1.For $\Re(s) > 0$, then

$$\psi(s) = \int_0^\infty \left[\frac{1}{e^t t} - \frac{e^{t(1-s)}}{e^t - 1}\right] du,$$

Where $\psi(s)$ denotes the digamma function and $\ln s$ denotes the natural logarithm.

Proof: I take $u + 1 = e^t$ and $du = e^t dt$ in Theorem 1; thus, we obtain

$$\psi(s) = \int_0^\infty \left[\frac{1}{e^{2t} t} - \frac{e^{-ts}}{e^t - 1}\right] e^t du$$

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$$= \int_0^\infty \left[\frac{1}{e^u u} - \frac{e^{u(1-s)}}{e^u - 1}\right] du .$$

Special Values: Using the Corollary 1, we calculate that

$$\psi\left(\frac{6}{5}\right) = 5 - \gamma - \frac{\pi}{2}\sqrt{1 + \frac{2}{\sqrt{5}}} - \frac{5\ln 5}{4} - \frac{\sqrt{5}}{2}\coth^{-1}(\sqrt{5}),$$

$$\psi\left(\frac{8}{7}\right) = 7 - \gamma - \ln 7 - \frac{\pi}{7}\sin\left(\frac{\pi}{7}\right) - \frac{3\pi}{7}\cos\left(\frac{\pi}{14}\right) - \frac{5\pi}{7}\cos\left(\frac{3\pi}{14}\right) - \sin\left(\frac{\pi}{14}\right)\ln\left(2 + 2\sin\left(\frac{\pi}{14}\right)\right)$$
$$+ \sin\left(\frac{3\pi}{14}\right)\ln\left(2 - 2\sin\left(\frac{3\pi}{14}\right)\right) - \cos\left(\frac{\pi}{7}\right)\ln\left(2 + 2\cos\left(\frac{\pi}{7}\right)\right),$$

$$\psi\left(\frac{10}{9}\right) = 9 - \gamma - \frac{5}{2}\ln 3 + \sin\left(\frac{\pi}{18}\right)\ln\left(2 - 2\sin\left(\frac{\pi}{18}\right)\right) - \cos\left(\frac{\pi}{9}\right)\ln\left(2\left(1 + \cos\left(\frac{\pi}{9}\right)\right)\right)$$
$$+ \cos\left(\frac{2\pi}{9}\right)\ln\left(2 - 2\cos\left(\frac{2\pi}{9}\right)\right) - \frac{\pi}{18}\left(3\sqrt{3} + 2\sin\left(\frac{\pi}{9}\right) + 14\sin\left(\frac{2\pi}{9}\right) + 10\cos\left(\frac{\pi}{18}\right)\right),$$

$$\psi\left(\frac{13}{12}\right) = 12 - \gamma - \frac{\sqrt{3}\,\pi}{2} - \pi - 3\ln 2 - \sqrt{3}\ln(2 + \sqrt{3}) - \frac{3\ln 3}{2},$$

$$\psi\left(\frac{15}{14}\right) = 14 - \gamma - 2\ln 2 - \ln 7 - 2\sin\left(\frac{\pi}{14}\right)\tanh^{-1}\left(\sin\left(\frac{\pi}{14}\right)\right) - 2\sin\left(\frac{3\pi}{14}\right)\tanh^{-1}\left(\sin\left(\frac{3\pi}{14}\right)\right)$$
$$- \pi\left(\sin\left(\frac{\pi}{7}\right) + \cos\left(\frac{\pi}{14}\right) + \cos\left(\frac{3\pi}{14}\right)\right) - 2\cos\left(\frac{\pi}{7}\right)\tanh^{-1}\left(\cos\left(\frac{\pi}{7}\right)\right),$$

$$\psi\left(\frac{16}{15}\right) = 15 - \gamma - \frac{\sqrt{3}\,\pi}{4} - \frac{\sqrt{15}\,\pi}{4} - \frac{\pi}{2}\sqrt{\frac{1}{2}(5 + \sqrt{5})} - \frac{3\ln 3}{2} - \frac{5\ln 5}{4} - \frac{\sqrt{5}}{2}\coth^{-1}(\sqrt{5})$$
$$- \frac{\sqrt{5}}{4}\coth^{-1}\left(\frac{3}{\sqrt{5}}\right) - \frac{1}{2}\sqrt{\frac{3(5 + \sqrt{5})}{2}}\coth^{-1}\left(\sqrt{\frac{5}{3} + \frac{2}{3\sqrt{5}}}\right)$$
$$- \frac{\sqrt{30 - 6\sqrt{5}}}{4}\coth^{-1}\left(\sqrt{\frac{5}{3} - \frac{2}{3\sqrt{5}}}\right).$$

REFERENCES

[1] http://functions.wolfram.com/GammaBetaErf/PolyGamma/07/01/01/