

# On the Inverse Cosine Function, $\pi$ and Babylonian Identity

<sup>1</sup>Edigles Guedes, <sup>2</sup>K. Raja Rama Gandhi

<sup>1</sup>Number Theorist, Brazil

<sup>2</sup>Resource person in Mathematics for Oxford University Press and Professor at BITS-izag

## ABSTRACT

We evaluate the constant  $\pi$  using the Babylonian identity and the inverse cosine or sine function.

## 1. INTRODUCTION

By means of the inverse cosine function and Babylonian identity, we demonstrated the identities following, among others:

$$\frac{\pi}{2} = \sin^{-1}\left(\frac{1}{3}\right) + \sin^{-1}\left(\frac{2\sqrt{2}}{3}\right),$$

$$\frac{\pi}{2} = \sin^{-1}\left(\frac{1}{4}\right) + \sin^{-1}\left(\frac{\sqrt{15}}{4}\right),$$

$$\frac{\pi}{2} = \sin^{-1}\left(\frac{1}{5}\right) + \sin^{-1}\left(\frac{2\sqrt{6}}{5}\right),$$

$$\frac{\pi}{2} = \sin^{-1}\left(\frac{1}{6}\right) + \sin^{-1}\left(\frac{\sqrt{35}}{6}\right),$$

$$\frac{\pi}{2} = \sin^{-1}\left(\frac{1}{7}\right) + \sin^{-1}\left(\frac{4\sqrt{3}}{7}\right),$$

$$\frac{\pi}{2} = \sin^{-1}\left(\frac{1}{8}\right) + \sin^{-1}\left(\frac{3\sqrt{7}}{8}\right),$$

$$\frac{\pi}{2} = \sin^{-1}\left(\frac{1}{9}\right) + \sin^{-1}\left(\frac{4\sqrt{5}}{9}\right),$$

$$\frac{\pi}{2} = \sin^{-1}\left(\frac{1}{10}\right) + \sin^{-1}\left(\frac{3\sqrt{11}}{10}\right),$$

and so on.

## 2. THEOREMS

Theorem 1. For  $0 < \operatorname{Re}(x) < 1$  and  $\operatorname{Im}(x) = 0$ , then

$$\cos^{-1}(x) = \frac{\pi}{2} - \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n}(n!)^2(2n+1)} x^{2n+1},$$

where  $\cos^{-1}(x)$  denotes the inverse cosine function.

*Proof.* In previous paper [1], we put  $\frac{a-b}{a+b} = t$  in (5), and encounter

$$(8) \quad \frac{1}{\sqrt{1-t^2}} = \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n}n!^2} t^{2n}.$$

we integrate (8) from  $z$  at 1 in  $t$ , then

$$(9) \quad \int_x^1 \frac{dt}{\sqrt{1-t^2}} = \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n}n!^2} \int_x^1 t^{2n} dt.$$

we calculate

$$(10) \quad \int_x^1 \frac{dt}{\sqrt{1-t^2}} = \cos^{-1}(x)$$

and

$$(11) \quad \int_x^1 t^{2n} dt = \frac{1-x^{2n+1}}{2n+1}.$$

From (9), (10) and (11), it follows

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$$(12) \quad \cos^{-1}(x) = \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n}(n!)^2(2n+1)} - \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n}(n!)^2(2n+1)} x^{2n+1}$$

$$= \frac{\pi}{2} - \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n}(n!)^2(2n+1)} x^{2n+1}. \square$$

Theorem 2. For  $x \in [0,1]$ , then

$$\cos^{-1}(x) = \frac{\pi}{2} - x\sqrt{1-x^2} - 4x \sum_{n=0}^{\infty} \frac{(2n)! n}{2^{2n}(n!)^2(4n^2-1)} x^{2n},$$

where  $\sin^{-1}(x)$  denotes the inverse sine function.

*Proof.* Using the identity

$$(13) \quad \frac{1}{2n+1} = \frac{4n}{4n^2-1} - \frac{1}{2n-1}$$

in the right-hand side of Theorem 1, we have

$$\cos^{-1}(x) = \frac{\pi}{2} - \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n}(n!)^2(2n+1)} x^{2n+1} = \frac{\pi}{2} - 4x \sum_{n=0}^{\infty} \frac{(2n)! n}{2^{2n}(n!)^2(4n^2-1)} x^{2n} + x \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n}(n!)^2(2n-1)} x^{2n}.$$

we calculate

$$(14) \quad \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n}(n!)^2(2n-1)} x^{2n} = -\sqrt{1-x^2}.$$

we substitute (14) into (13), and obtain

$$\cos^{-1}(x) = \frac{\pi}{2} - x\sqrt{1-x^2} - 4x \sum_{n=0}^{\infty} \frac{(2n)! n}{2^{2n}(n!)^2(4n^2-1)} x^{2n}. \square$$

Theorem 3. For  $x \in [0, \pi]$ , then

$$\cos^{-1}(\sin x) = \frac{\pi - \sin(2x)}{2} - 4 \sin x \sum_{n=1}^{\infty} \frac{(2n)! n}{2^{2n}(n!)^2(4n^2-1)} (\sin x)^{2n},$$

where  $\cos^{-1}(x)$  denotes the inverse sine function,  $\sin(x)$  denotes the sine function and  $\cos(x)$  denotes the cosine function.

*Proof.* Let  $x = \sin x$  in Theorem 2

$$(15) \quad 4 \sum_{n=1}^{\infty} \frac{(2n)! n}{2^{2n}(n!)^2(4n^2-1)} (\sin x)^{2n+1} = \frac{\pi}{2} - \frac{\sin(2x)}{2} - \cos^{-1}(\sin x) \Rightarrow$$

$$\cos^{-1}(\sin x) = \frac{\pi - \sin(2x)}{2} - 4 \sin x \sum_{n=1}^{\infty} \frac{(2n)! n}{2^{2n}(n!)^2(4n^2-1)} (\sin x)^{2n}. \square$$

*Explicit Calculations.* we leave for the reader:

$$\sum_{n=1}^{\infty} \frac{(2n)! n}{2^{2n}(n!)^2(4n^2-1)} \left(\frac{\sqrt{2+\sqrt{2}}}{2}\right)^{2n} = \frac{3\pi - 2\sqrt{2}}{16\sqrt{2+\sqrt{2}}}$$

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$$\sum_{n=1}^{\infty} \frac{(2n)! n}{2^{2n}(n!)^2(4n^2 - 1)} \left( \sqrt{\frac{5}{8} + \frac{\sqrt{5}}{8}} \right)^{2n} = \frac{8\sqrt{2}\pi - 5\sqrt{5 - \sqrt{5}}}{40\sqrt{5 + \sqrt{5}}},$$

$$\sum_{n=1}^{\infty} \frac{(2n)! n}{2^{2n}(n!)^2(4n^2 - 1)} \left( \frac{\sqrt{2 + \sqrt{2 + \sqrt{2}}}}{2} \right)^{2n} = \frac{7\pi - 4\sqrt{2 - \sqrt{2}}}{32\sqrt{2 + \sqrt{2 + \sqrt{2}}}}$$

$$\sum_{n=1}^{\infty} \frac{(2n)! n}{2^{2n}(n!)^2(4n^2 - 1)} \left( \frac{1 + \sqrt{3}}{2\sqrt{2}} \right)^{2n} = \frac{\sqrt{2}(5\pi - 3)}{24(1 + \sqrt{3})},$$

$$\sum_{k=1}^{\infty} \frac{(2n)! n}{2^{2n}(n!)^2(4n^2 - 1)} \left( \frac{1 + \sqrt{5}}{4\sqrt{2}} + \frac{\sqrt{5 - \sqrt{5}}}{4} \right)^{2n} = \frac{18\sqrt{2}\pi - 5(\sqrt{10} - \sqrt{2})}{40(1 + \sqrt{5} + \sqrt{10 - 2\sqrt{5}})},$$

$$\sum_{n=1}^{\infty} \frac{(2n)! n}{2^{2n}(n!)^2(4n^2 - 1)} \left( \frac{\sqrt{5 + \sqrt{5}}}{4} - \frac{1 - \sqrt{5}}{4\sqrt{2}} \right)^{2n} = \frac{14\pi - 5(1 + \sqrt{5})}{20(2\sqrt{5 + \sqrt{5}} + \sqrt{10} - \sqrt{2})}$$

and

$$\sum_{n=1}^{\infty} \frac{(2n)! n}{2^{2n}(n!)^2(4n^2 - 1)} \left( \frac{\sqrt{3}}{2} \right)^{2n} = \frac{4\pi - 3\sqrt{3}}{24\sqrt{3}},$$

consequently,

$$\pi = \frac{2\sqrt{2}}{3} + \frac{16\sqrt{2 + \sqrt{2}}}{3} \sum_{n=1}^{\infty} \frac{(2n)! n}{2^{2n}(n!)^2(4n^2 - 1)} \left( \frac{\sqrt{2 + \sqrt{2}}}{2} \right)^{2n},$$

$$\pi = \frac{5}{8} \sqrt{\frac{5}{2} - \frac{\sqrt{5}}{2}} + 5 \sqrt{\frac{5}{2} + \frac{\sqrt{5}}{2}} \sum_{n=1}^{\infty} \frac{(2n)! n}{2^{2n}(n!)^2(4n^2 - 1)} \left( \sqrt{\frac{5}{8} + \frac{\sqrt{5}}{8}} \right)^{2n},$$

$$\pi = \frac{4}{7} \sqrt{2 - \sqrt{2}} + \frac{32}{7} \sqrt{2 + \sqrt{2 + \sqrt{2}}} \sum_{n=1}^{\infty} \frac{(2n)! n}{2^{2n}(n!)^2(4n^2 - 1)} \left( \frac{\sqrt{2 + \sqrt{2 + \sqrt{2}}}}{2} \right)^{2n},$$

$$\pi = \frac{3}{5} + \frac{24(1 + \sqrt{3})}{5\sqrt{2}} \sum_{n=1}^{\infty} \frac{(2n)! n}{2^{2n}(n!)^2(4n^2 - 1)} \left( \frac{1 + \sqrt{3}}{2\sqrt{2}} \right)^{2n},$$

$$\pi = \frac{5(\sqrt{5} - 1)}{18} + \frac{20(1 + \sqrt{5} + \sqrt{10 - 2\sqrt{5}})}{9\sqrt{2}} \sum_{n=1}^{\infty} \frac{(2n)! n}{2^{2n}(n!)^2(4n^2 - 1)} \left( \frac{1 + \sqrt{5}}{4\sqrt{2}} + \frac{\sqrt{5 - \sqrt{5}}}{4} \right)^{2n},$$

$$\pi = \frac{5(1 + \sqrt{5})}{14} + \frac{10}{7} \left( 2\sqrt{5 + \sqrt{5}} + \sqrt{10} - \sqrt{2} \right) \sum_{k=1}^{\infty} \frac{(2n)! n}{2^{2n}(n!)^2(4n^2 - 1)} \left( \frac{\sqrt{5 + \sqrt{5}}}{4} - \frac{1 - \sqrt{5}}{4\sqrt{2}} \right)^{2n}$$

and

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$$\pi = \frac{3\sqrt{3}}{4} + 6\sqrt{3} \sum_{n=1}^{\infty} \frac{(2n)! n}{2^{2n} (n!)^2 (4n^2 - 1)} \left(\frac{\sqrt{3}}{2}\right)^{2n},$$

just as in the previous paper [1].

Corollary 1. we have

$$\pi^2 = 4 + 16\sqrt{\pi} \sum_{n=1}^{\infty} \frac{(2n)! n}{2^{2n} n! (4n^2 - 1) \Gamma\left(n + \frac{3}{2}\right)} = 4 + 32 \sum_{n=1}^{\infty} \frac{n}{(2n-1)(2n+1)^2}.$$

*Proof.* we integrate the Theorem 3 from 0 at  $\frac{\pi}{2}$  with respect to  $x$

$$(16) \quad \int_0^{\frac{\pi}{2}} \cos^{-1}(\sin x) dx = \int_0^{\frac{\pi}{2}} \frac{\pi - \sin(2x)}{2} dx - 4 \sum_{n=1}^{\infty} \frac{(2n)! n}{2^{2n} (n!)^2 (4n^2 - 1)} \int_0^{\frac{\pi}{2}} (\sin x)^{2n+1} dx.$$

we calculate

$$(17) \quad \int_0^{\frac{\pi}{2}} \cos^{-1}(\sin x) dx = \frac{\pi^2}{8},$$

$$(18) \quad \int_0^{\frac{\pi}{2}} \frac{\pi - \sin(2x)}{2} dx = \frac{\pi^2 - 2}{4},$$

and

$$(19) \quad \int_0^{\frac{\pi}{2}} (\sin x)^{2k+1} dx = \frac{\sqrt{\pi} \Gamma(k+1)}{2\Gamma\left(k + \frac{3}{2}\right)},$$

for  $k > -1$ .

From (16), (17), (18) and (19), the proof follows.  $\square$

### 3. NOTE ON PREVIOUS PAPER

In [1], we may expand the summation and encounter the explicit evaluation of the inverse sine function:

$$\sin^{-1}\left(\frac{\sqrt{2+\sqrt{2}}}{2}\right) = \frac{3\pi + 4\sqrt{3-2\sqrt{2}} + 2\sqrt{6-4\sqrt{2}} - 2\sqrt{2}}{8},$$

$$\sin^{-1}\left(\sqrt{\frac{5+\sqrt{5}}{8}}\right) = \frac{2\pi}{5} + \frac{\sqrt{5-2\sqrt{5}} + \sqrt{25-10\sqrt{5}}}{8} - \frac{1}{4} \sqrt{\frac{10}{5+\sqrt{5}}},$$

$$\sin^{-1}\left(\frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2}\right) = \frac{7\pi}{16} + \frac{1}{4} \sqrt{\frac{(2+\sqrt{2})(2-\sqrt{2+\sqrt{2}})}{2+\sqrt{2+\sqrt{2}}}} + \frac{1}{2} \sqrt{\frac{2-\sqrt{2+\sqrt{2}}}{2+\sqrt{2+\sqrt{2}}}} - \frac{\sqrt{2-\sqrt{2}}}{4},$$

$$\sin^{-1}\left(\frac{1+\sqrt{3}}{2\sqrt{2}}\right) = \frac{5\pi-3}{12} + \frac{1+\sqrt{3}}{4} \sqrt{\frac{2-\sqrt{3}}{2}},$$

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$$\begin{aligned} \sin^{-1}\left(\frac{2\sqrt{5-\sqrt{5}}+\sqrt{2}+\sqrt{10}}{8}\right) &= \frac{9\pi}{20} + \frac{\sqrt{16-2\sqrt{10-2\sqrt{5}}-2\sqrt{50-10\sqrt{5}}}}{32} \\ &+ \frac{\sqrt{80-10\sqrt{10-2\sqrt{5}}-10\sqrt{50-10\sqrt{5}}}}{32} + \frac{\sqrt{10-2\sqrt{5}-\sqrt{50-10\sqrt{5}}}}{8} \\ &- \frac{4+\sqrt{50-10\sqrt{5}}-\sqrt{10-2\sqrt{5}}}{8(\sqrt{10-2\sqrt{5}}+\sqrt{5}+1)}, \end{aligned}$$

$$\sin^{-1}\left(\frac{2\sqrt{5+\sqrt{5}}-\sqrt{2}+\sqrt{10}}{8}\right) =$$

$$\begin{aligned} \frac{7\pi}{20} + \frac{\sqrt{10+2\sqrt{5}-\sqrt{50+10\sqrt{5}}}}{8} + \frac{5\sqrt{80+10\sqrt{10+2\sqrt{5}}-10\sqrt{50+10\sqrt{5}}}}{160} - \frac{5\sqrt{16+2\sqrt{10+2\sqrt{5}}-2\sqrt{50+10\sqrt{5}}}}{160} \\ - \frac{\sqrt{5}}{8} - \frac{1}{8}. \end{aligned}$$

Theorem 3.1. For  $x \in [0,1]$ , then

$$\cos^{-1}(x) = x\sqrt{1-x^2} + 4\sqrt{1-x^2} \sum_{n=1}^{\infty} \frac{(2n)!n}{2^{2n}(n!)^2(4n^2-1)} (1-x^2)^n$$

or

$$\sum_{n=1}^{\infty} \frac{(2n)!n}{2^{2n}(n!)^2(4n^2-1)} (1-x^2)^n = \frac{\cos^{-1}(x) - x\sqrt{1-x^2}}{4\sqrt{1-x^2}},$$

where  $\cos^{-1}(x)$  denotes the inverse cosine function.

*Proof.* In previous paper [1], we proved that

$$(3.1) \quad \sin^{-1}(x) = x\sqrt{1-x^2} + 4x \sum_{n=1}^{\infty} \frac{(2n)!n}{2^{2n}(n!)^2(4n^2-1)} x^{2n}.$$

we set  $\sin x$  by  $x$  into (3.1)

$$(3.2) \quad x = \frac{\sin(2x)}{2} + 4 \sin x \sum_{n=1}^{\infty} \frac{(2n)!n}{2^{2n}(n!)^2(4n^2-1)} (\sin x)^{2n}.$$

we replace  $\cos^{-1} x$  by  $x$  into (3.2)

$$\begin{aligned} (3.3) \quad \cos^{-1} x &= \frac{\sin(2 \cos^{-1} x)}{2} + 4 \sin(\cos^{-1} x) \sum_{n=1}^{\infty} \frac{(2n)!n}{2^{2n}(n!)^2(4n^2-1)} [\sin(\cos^{-1} x)]^{2n} \\ &= x\sqrt{1-x^2} + 4\sqrt{1-x^2} \sum_{n=1}^{\infty} \frac{(2n)!n}{2^{2n}(n!)^2(4n^2-1)} (\sqrt{1-x^2})^{2n} \end{aligned}$$

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$$= x\sqrt{1-x^2} + 4\sqrt{1-x^2} \sum_{n=1}^{\infty} \frac{(2n)!n}{2^{2n}(n!)^2(4n^2-1)} (1-x^2)^n. \square$$

**Explicit Evaluation.** Let  $x = \frac{1+\sqrt{5}}{4}$  in Theorem 3.1

$$\sum_{n=1}^{\infty} \frac{(2n)!n}{2^{2n}(n!)^2(4n^2-1)} \left(\frac{3}{4}\right)^n = \frac{4\pi - 3\sqrt{3}}{24\sqrt{3}},$$

and so on. we may use the Theorem 3.1 and expansion of the summation to obtain the identities

$$\cos^{-1}\left(\frac{1}{3}\right) = \sin^{-1}\left(\frac{2\sqrt{2}}{3}\right),$$

$$\cos^{-1}\left(\frac{1}{4}\right) = \sin^{-1}\left(\frac{\sqrt{15}}{4}\right),$$

$$\cos^{-1}\left(\frac{1}{5}\right) = \sin^{-1}\left(\frac{2\sqrt{6}}{5}\right),$$

$$\cos^{-1}\left(\frac{1}{6}\right) = \sin^{-1}\left(\frac{\sqrt{35}}{6}\right),$$

$$\cos^{-1}\left(\frac{1}{7}\right) = \sin^{-1}\left(\frac{4\sqrt{3}}{7}\right),$$

$$\cos^{-1}\left(\frac{1}{8}\right) = \sin^{-1}\left(\frac{3\sqrt{7}}{8}\right),$$

$$\cos^{-1}\left(\frac{1}{9}\right) = \sin^{-1}\left(\frac{4\sqrt{5}}{9}\right),$$

$$\cos^{-1}\left(\frac{1}{10}\right) = \sin^{-1}\left(\frac{3\sqrt{11}}{10}\right),$$

and so on. Using the identity  $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$  in previous identities, we have

$$\frac{\pi}{2} = \sin^{-1}\left(\frac{1}{3}\right) + \sin^{-1}\left(\frac{2\sqrt{2}}{3}\right),$$

$$\frac{\pi}{2} = \sin^{-1}\left(\frac{1}{4}\right) + \sin^{-1}\left(\frac{\sqrt{15}}{4}\right),$$

$$\frac{\pi}{2} = \sin^{-1}\left(\frac{1}{5}\right) + \sin^{-1}\left(\frac{2\sqrt{6}}{5}\right),$$

$$\frac{\pi}{2} = \sin^{-1}\left(\frac{1}{6}\right) + \sin^{-1}\left(\frac{\sqrt{35}}{6}\right),$$

$$\frac{\pi}{2} = \sin^{-1}\left(\frac{1}{7}\right) + \sin^{-1}\left(\frac{4\sqrt{3}}{7}\right),$$

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$$\frac{\pi}{2} = \sin^{-1}\left(\frac{1}{8}\right) + \sin^{-1}\left(\frac{3\sqrt{7}}{8}\right),$$

$$\frac{\pi}{2} = \sin^{-1}\left(\frac{1}{9}\right) + \sin^{-1}\left(\frac{4\sqrt{5}}{9}\right),$$

$$\frac{\pi}{2} = \sin^{-1}\left(\frac{1}{10}\right) + \sin^{-1}\left(\frac{3\sqrt{11}}{10}\right),$$

and so on; as well as

$$\frac{\pi}{2} = \cos^{-1}\left(\frac{1}{3}\right) + \cos^{-1}\left(\frac{2\sqrt{2}}{3}\right),$$

$$\frac{\pi}{2} = \cos^{-1}\left(\frac{1}{4}\right) + \cos^{-1}\left(\frac{\sqrt{15}}{4}\right),$$

$$\frac{\pi}{2} = \cos^{-1}\left(\frac{1}{5}\right) + \cos^{-1}\left(\frac{2\sqrt{6}}{5}\right),$$

$$\frac{\pi}{2} = \cos^{-1}\left(\frac{1}{6}\right) + \cos^{-1}\left(\frac{\sqrt{35}}{6}\right),$$

$$\frac{\pi}{2} = \cos^{-1}\left(\frac{1}{7}\right) + \cos^{-1}\left(\frac{4\sqrt{3}}{7}\right),$$

$$\frac{\pi}{2} = \cos^{-1}\left(\frac{1}{8}\right) + \cos^{-1}\left(\frac{3\sqrt{7}}{8}\right),$$

$$\frac{\pi}{2} = \cos^{-1}\left(\frac{1}{9}\right) + \cos^{-1}\left(\frac{4\sqrt{5}}{9}\right),$$

$$\frac{\pi}{2} = \cos^{-1}\left(\frac{1}{10}\right) + \cos^{-1}\left(\frac{3\sqrt{11}}{10}\right),$$

and so on.

Theorem 3.2. For  $x \in [0,1]$ , then

$$\sin^{-1}(x) = x\sqrt{1-x^2} + 2x^3 \sum_{n=0}^{\infty} \frac{(2n+2)!}{2^{2n+1}(2n+3)(2n+1)(n+1)} \frac{x^{2n}}{(n!)^2}$$

or

$$\sum_{n=0}^{\infty} \frac{(2n+2)!}{2^{2n+1}(2n+3)(2n+1)(n+1)} \frac{x^{2n}}{(n!)^2} = \frac{\sin^{-1}(x) - x\sqrt{1-x^2}}{2x^3},$$

where  $\sin^{-1}(x)$  denotes the inverse sine function.

*Proof.* In previous paper [1], we proved that

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$$(3.4) \quad \sin^{-1}(x) = x\sqrt{1-x^2} + 4x \sum_{n=1}^{\infty} \frac{(2n)!n}{2^{2n}(n!)^2(4n^2-1)} x^{2n}.$$

*Proof.* In previous paper [1], we proved that

$$(3.5) \quad \sin^{-1}(x) = x\sqrt{1-x^2} + 4x \sum_{n=1}^{\infty} \frac{(2n)!n}{2^{2n}(n!)^2(4n^2-1)} x^{2n}.$$

we calculate the identity

$$(3.6) \quad \frac{1}{4n^2-1} = \int_0^1 \frac{1}{(2n+2v-1)^2} dv.$$

we set (3.6) into (3.5)

$$(3.7) \quad \begin{aligned} \sin^{-1}(x) &= x\sqrt{1-x^2} + 4x \sum_{n=1}^{\infty} \frac{(2n)!n}{2^{2n}(n!)^2} \int_0^1 \frac{1}{(2n+2v-1)^2} dv x^{2n} \\ &= x\sqrt{1-x^2} + 4x \int_0^1 \sum_{n=1}^{\infty} \frac{(2n)!n}{2^{2n}(n!)^2(2n+2v-1)^2} x^{2n} dv \\ &= x\sqrt{1-x^2} + 2x^3 \int_0^1 \frac{1}{(2v+1)^2} {}_3F_2 \left( \begin{matrix} \frac{3}{2}, v+\frac{1}{2}, v+\frac{1}{2} \\ v+\frac{3}{2}, v+\frac{3}{2} \end{matrix} \middle| x^2 \right) dv \\ &= x\sqrt{1-x^2} + 2x^3 \int_0^1 \frac{1}{(2v+1)^2} \sum_{n=0}^{\infty} \frac{\left(\frac{3}{2}\right)_n \left(v+\frac{1}{2}\right)_n \left(v+\frac{1}{2}\right)_n}{\left(v+\frac{3}{2}\right)_n \left(v+\frac{3}{2}\right)_n} x^{2n} dv \\ &= x\sqrt{1-x^2} + 2x^3 \sum_{n=0}^{\infty} \left(\frac{3}{2}\right)_n \int_0^1 \frac{\left(v+\frac{1}{2}\right)_n \left(v+\frac{1}{2}\right)_n}{(2v+1)^2 \left(v+\frac{3}{2}\right)_n \left(v+\frac{3}{2}\right)_n} dv \frac{x^{2n}}{n!} \\ &= x\sqrt{1-x^2} + 2x^3 \sum_{n=0}^{\infty} \frac{\left(\frac{3}{2}\right)_n}{4n^2+8n+3} \frac{x^{2n}}{n!} \\ &= x\sqrt{1-x^2} + 2x^3 \sum_{n=0}^{\infty} \frac{1}{4n^2+8n+3} \frac{(2n+2)!}{2^{2n+1}(n+1)!} \frac{x^{2n}}{n!} \\ &= x\sqrt{1-x^2} + 2x^3 \sum_{n=0}^{\infty} \frac{(2n+2)!}{2^{2n+1}(2n+3)(2n+1)(n+1)} \frac{x^{2n}}{(n!)^2}. \square \end{aligned}$$

## REFERENCES

[1] <http://www.bmsa.us/admin/uploads/ad5AMh.pdf>