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An Integral Representation for the Digamma Function Arising from Abel-Plana Formula

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ABSTRACT

We proved a new integral representation for the digamma function arising from Abel-Plana formula

Keywords: *Integral, digamma function*

1. INTRODUCTION

Using an integral representation of the natural logarithm and the Abel-Plana formula, we demonstrated that:

$$\psi(z) = -\gamma + \ln z + \frac{1}{2}\left(1 - \frac{1}{z}\right) + 2(z^2 - 1)\int_0^\infty \frac{x}{(x^2+1)(x^2+z^2)(e^{2\pi x}-1)}dx.$$

2. THEOREM

Theorem 1: For $\Re(z) > 0$, then,

$$\psi(z) = -\gamma + \ln z + \frac{1}{2}\left(1 - \frac{1}{z}\right) + 2(z^2 - 1)\int_0^\infty \frac{x}{(x^2+1)(x^2+z^2)(e^{2\pi x}-1)}dx,$$

Where $\psi(z)$ denotes the digamma function and $\ln z$ denotes the logarithm function.

Proof: In the next paper [1], for $\Re(z) > 0$, wewill prove that

$$\ln z = (z-1)\int_0^\infty \frac{1}{(t+z)(t+1)}dt. \qquad (1)$$

The most frequently used form of the Abel-Plana formula is the following [2]

$$\sum_{n=0}^{\infty} f(n) = \int_0^\infty f(x)\,dx + \frac{1}{2}f(0) + i\int_0^\infty \frac{f(ix) - f(-ix)}{e^{2\pi x}-1}dx. \qquad (2)$$

It is applicable to functions satisfying the following convergence condition

$$\lim_{y\to\infty} e^{-2\pi|y|}\,|f(x+iy)| = 0 \qquad (3)$$

Uniformly on any finite interval of x.

By the Abel-Plana Formula, it follows that

$$\sum_{n=0}^{\infty}\frac{1}{(n+z)(n+1)} = \int_0^{\infty}\frac{1}{(x+z)(x+1)}dx + \frac{1}{2z} + i\int_0^{\infty}\frac{\frac{1}{(z+ix)(1+ix)} - \frac{1}{(z-ix)(1-ix)}}{e^{2\pi x}-1}dx. \quad (4)$$

We substitute (1) in (4) and expand the left hand side to obtain

$$\sum_{n=0}^{\infty}\frac{1}{(n+z)(n+1)} = \frac{\ln z}{z-1} + \frac{1}{2z} + 2\int_0^{\infty}\frac{x(z+1)}{(x^2+1)(x^2+z^2)(e^{2\pi x}-1)}dx. \quad (5)$$

Now, we expand the right hand side of the (5)

$$\frac{\psi(z)+\gamma}{z-1} = \frac{\ln z}{z-1} + \frac{1}{2z} + 2(z+1)\int_0^{\infty}\frac{x}{(x^2+1)(x^2+z^2)(e^{2\pi x}-1)}dx. \quad (6)$$

Multiply (6) by $z-1$, then

$$\psi(z) = -\gamma + \ln z + \frac{1}{2}\left(1-\frac{1}{z}\right) + 2(z^2-1)\int_0^{\infty}\frac{x}{(x^2+1)(x^2+z^2)(e^{2\pi x}-1)}dx.$$

REFERENCES

[1] Guedes, Edigles and Prof. Dr. K. Raja Rama Gandhi, On the Natural Logarithm and its Applications, 2014, to appear.

[2] Saharian, Aram A., Generalized Abel-Plana Formula as a Renormalization tool in Quantum field Theory with Boundaries, arXiv:hep-th/0609093.