Steady Plane Coquette Flow of Viscous Incompressible Fluid between Two Porous Parallel Plates through Porous Medium

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ABSTRACT

In this paper, we have investigated the steady plane Coquette flow of viscous incompressible fluid between two porous parallel plates through porous medium. We have investigated the velocity, average velocity, shearing stress, skin frictions, the volumetric flow, drag coefficients and streamlines.

Keywords: Steady Coquette flow, viscous parallel plates, incompressible fluid and porous medium

NOMENCLATURE

\( u \) = Velocity component along \( x \) – axis
\( v \) = Velocity component along \( y \) – axis
\( t \) = the time
\( P \) = the fluid pressure
\( \rho \) = the density of fluid
\( K \) = the thermal conductivity of the fluid
\( \mu \) = Coefficient of viscosity
\( \nu \) = Kinematic viscosity
\( Q \) = the volumetric flow

1. INTRODUCTION


2. FORMULATION OF PROBLEM

Let us consider two infinite porous plates AB & CD separated by a distance 2h. The fluid enters in y - direction. The velocity component along x – axis is a function of y only. The motion of incompressible fluid is in two dimension and is steady then

\[ u = u(y), \quad w = 0 \quad \text{and} \quad \frac{\partial}{\partial t} = 0 \]

The equation of continuity for incompressible fluid

\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad \text{and \ put \ } w = 0, \quad \frac{\partial u}{\partial x} = 0, \quad \text{and} \quad \Rightarrow \frac{\partial v}{\partial y} = 0 \]

\( v \) is independent of \( y \) but motion along \( y \) – axis. So we can say \( v \) is constant velocity i.e. \( v = v_0 \)

or The fluid enters the flow region through one plate at the same constant velocity \( v_0 \)

Also Navier - Stoke's equations for incompressible fluid in the absence of body force when flow is steady
3. SOLUTION OF THE PROBLEM

Equation (2) Shows that the pressure does not depend on \( y \) hence \( p \) is a function of \( x \) only and so (1) reduces to

\[
\frac{dp}{dx} = \rho \left[ v \frac{d^2u}{dy^2} - v_0 \frac{du}{dy} + \frac{vu}{K} \right]
\]

Where \( \frac{dp}{dx} = \text{Constant} = -P \)

\[
\Rightarrow \frac{d^2u}{dy^2} - \frac{v_0}{v} \frac{du}{dy} + \frac{u}{K} = -\frac{P}{\rho v}
\]

\[
\Rightarrow \left(D^2 - \frac{v_0}{v} D + \frac{1}{K}\right)u = -\frac{P}{\rho v}
\]

A.E. \( m^2 - \frac{v_0}{v} m + \frac{1}{K} = 0 \)

\[
\frac{PK}{\mu} e^{\frac{v_0 h}{2\mu}} = C_1\cosh{A}h - C_2\sinh{A}h \quad \& \quad \left( U + \frac{PK}{\mu} \right) e^{\frac{v_0 h}{2\mu}} = C_1\cosh{A}h + C_2\sinh{A}h
\]

\[
C_1 = \frac{1}{2\cosh{A}h} \left[ \left(U + \frac{PK}{\mu} \right) e^{\frac{v_0 h}{2\mu}} + \frac{PK}{\mu} e^{\frac{v_0 h}{2\mu}} \right] \quad \& \quad C_2 = \frac{1}{2\sinh{A}h} \left[ \left(U + \frac{PK}{\mu} \right) e^{-\frac{v_0 h}{2\mu}} - \frac{PK}{\mu} e^{-\frac{v_0 h}{2\mu}} \right]
\]

\[
u(y) = \frac{v_0}{2\cosh{A}h} \cosh{A}y \left[ \left(U + \frac{PK}{\mu} \right) e^{\frac{v_0 h}{2\mu}} + \frac{PK}{\mu} e^{\frac{v_0 h}{2\mu}} \right] + \frac{v_0}{2\sinh{A}h} \sinh{A}y \left[ \left(U + \frac{PK}{\mu} \right) e^{-\frac{v_0 h}{2\mu}} - \frac{PK}{\mu} e^{-\frac{v_0 h}{2\mu}} \right]
\]

\[
u(y) = \left(U + \frac{PK}{\mu} \right) e^{\frac{v_0}{2\mu}(y - h)} \sinh{A}(y - h) - \frac{PK}{\mu} e^{\frac{v_0}{2\mu}(y + h)} \sinh{A}(y + h) - \frac{PK}{\mu} e^{\frac{v_0}{2\mu}y} \sinh{A}h
\]
\[ u(y) = \frac{1}{\sinh 2Ah} \left[ U + \frac{PK}{\mu} e^{2\alpha(y-h)} \sinh A(y + h) - \frac{PK}{\mu} e^{2\alpha(y-h)} \sinh A(y - h) \right] - \frac{PK}{\mu} \quad (5) \]

**Plane Coquette flow:** In this case \( P = 0 \)

\[ u(y) = \frac{1}{\sinh 2Ah} \left[ U e^{2\alpha(y-h)} \sinh A(y + h) \right] \quad \ldots \ldots \quad (6) \]

The shearing stress at any point

\[ \sigma_{xy} = \frac{\mu}{h} \frac{du}{dy} = \frac{\mu U}{\sinh 2Ah} \left[ \frac{V_0}{2\nu} e^{2\alpha(y-h)} \sinh A(y + h) + A e^{2\alpha(y-h)} \cosh A(y + h) \right] \]

\[ = \frac{\mu U e^{2\alpha(y-h)}}{\sinh 2Ah} \left[ \frac{V_0}{2\nu} \sinh A(y + h) + ACosh A(y + h) \right] \quad \ldots \ldots \quad (7) \]

The skin frictions at Lower and Upper plate is given by

\[ (\sigma_{xy})_{y=h} = \frac{\mu U \theta}{\sinh 2Ah} \left[ \frac{V_0}{2\nu} - \sinh 2Ah + ACosh 2Ah \right] = \mu U \left[ \frac{V_0}{2\nu} + ACoth 2Ah \right] \quad \ldots \ldots \quad (8) \]

\[ (\sigma_{xy})_{y=-h} = \frac{\mu U e^{-\frac{\theta}{\nu}}}{\sinh 2Ah} \frac{A}{\sinh 2Ah} \quad \ldots \ldots \quad (9) \]

The average velocity distribution in plane coquette flow:

\[ (u)_{av} = \frac{1}{2h} \int_{-h}^{h} u(y) \, dy = \frac{1}{2h} \int_{-h}^{h} U \frac{V_0}{2\nu} e^{2\alpha(y-h)} \sinh A(y + h) \, dy = \frac{U}{2h \sinh 2Ah} \int_{-h}^{h} \frac{V_0}{2\nu} e^{2\alpha(y-h)} \left( e^{A(y+h)} - e^{-A(y+h)} \right) \, dy \]

\[ = \frac{U}{4h \sinh 2Ah} \int_{-h}^{h} \left( \frac{V_0}{2\nu} \left( e^{2\alpha(y-h) + A(y+h)} - e^{2\alpha(y-h) - A(y+h)} \right) \right) \, dy \]

\[ = \frac{U}{4h \sinh 2Ah} \left[ \frac{V_0}{2\nu} \left( e^{2Ah} - e^{-\frac{V_0}{2\nu}} \right) - \frac{V_0}{2\nu} \left( e^{-2Ah} - e^{-\frac{V_0}{2\nu}} \right) \right] \]

\[ = \frac{U}{4h \sinh 2Ah} \left[ \left( \frac{V_0}{2\nu} - A \right) \left( e^{2Ah} - e^{-\frac{V_0}{2\nu}} \right) - \left( \frac{V_0}{2\nu} + A \right) \left( e^{-2Ah} - e^{-\frac{V_0}{2\nu}} \right) \right] \]

\[ \left\{ \frac{V_0^2}{4\nu^2} - A^2 \right\} \]
Since \[
\left(\frac{v_0}{2\nu}\right)^2 - \frac{1}{K} = A \Rightarrow \left(\frac{v_0}{2\nu}\right)^2 - A^2 = \frac{1}{K}
\]

\[
(u)_{av} = \frac{UK}{4h \text{Sinh} 2Ah} \left[ \frac{v_0}{2\nu} \left\{ e^{2Ah} - e^{-\frac{v_0 h}{\nu}} - e^{-2Ah} + e^{-\frac{v_0 h}{\nu}} \right\} - A \left\{ e^{2Ah} - e^{-\frac{v_0 h}{\nu}} + e^{-2Ah} - e^{-\frac{v_0 h}{\nu}} \right\} \right]
\]

\[
= \frac{UK}{4h \text{Sinh} 2Ah} \left[ \frac{v_0}{2\nu} \text{Sinh} 2Ah - A \left\{ 2 \text{Cosh} 2Ah - 2e^{-\frac{v_0 h}{\nu}} \right\} \right]
\]

\[
= \frac{UK}{2h \text{Sinh} 2Ah} \left[ \frac{v_0}{2\nu} \text{Sinh} 2Ah - A \left\{ \text{Cosh} 2Ah - e^{-\frac{v_0 h}{\nu}} \right\} \right] \quad \text{......... (10)}
\]

The volumetric flow: \[
Q = 2h u_{av}
\]

\[
= \frac{UK}{\text{Sinh} 2Ah} \left[ \frac{v_0}{2\nu} \text{Sinh} 2Ah - A \left\{ \text{Cosh} 2Ah - e^{-\frac{v_0 h}{\nu}} \right\} \right] \quad \text{......... (11)}
\]

The drag coefficients:

\[
C_f \quad \text{&} \quad C_f' \quad \text{at} \quad y = h \quad \text{&} \quad y = -h
\]

\[
C_f = \frac{\left(\sigma_{xy}\right)_{y=h}}{\frac{1}{2} \rho u_{av}^2} = \frac{\mu U}{\text{Sinh} 2Ah} \left[ \frac{v_0}{2\nu} \text{Sinh} 2Ah + A\text{Cosh} 2Ah \right]
\]

\[
= \frac{8h^2 \mu \text{Sinh} 2Ah \left[ \frac{v_0}{2\nu} \text{Sinh} 2Ah + A\text{Cosh} 2Ah \right]}{\rho UK^2 \left[ \frac{v_0}{2\nu} \text{Sinh} 2Ah - A\text{Cosh} 2Ah + A\nu \right]^2} \quad \text{......... (12)}
\]

\[
C_f' = \frac{\left(\sigma_{xy}\right)_{y=-h}}{\frac{1}{2} \rho u_{av}^2} = \frac{\mu U e^{-\frac{v_0 h}{\nu}}}{\text{Sinh} 2Ah} \cdot \frac{8h^2 \text{Sinh}^2 2Ah}{\rho U^2 K^2 \left[ \frac{v_0}{2\nu} \text{Sinh} 2Ah - A\text{Cosh} 2Ah + A\nu \right]^2}
\]

\[
= \frac{8\mu h^2 A e^{-\frac{v_0 h}{\nu}} \text{Sinh} 2Ah}{\rho UK^2 \left[ \frac{v_0}{2\nu} \text{Sinh} 2Ah - A\text{Cosh} 2Ah + A\nu \right]^2} \quad \text{......... (13)}
\]
Equation of stream line in the plane coquette flow:

\[
\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}, \quad q = ui + vj + wk
\]

\[
\frac{dx}{U} = \frac{dy}{v_0 e^{2\nu(y-h)}} = \frac{dz}{v_0 e^{2\nu(y + h)}}
\]

\[
\text{Sinh} \ 2Ah
\]

Taking first two equations:

\[
\frac{v_0 \text{Sinh} \ 2Ah}{U} \int dx = \int e^{2\nu(y-h)} \text{Sinh} \ A(y + h) \ dy + C_1
\]

\[
x = \frac{v_0 \text{Sinh} \ 2Ah}{U} \left(1 - \frac{1}{2} \int \left\{e^{2\nu(y-h)} + A(y + h) - e^{2\nu(y-h)} A(y + h)\right\} dy\right) = C_1
\]

First stream line

\[
\Rightarrow \frac{v_0 \text{Sinh} \ 2Ah}{U} x - \frac{K}{2} \left(\left(\frac{v_0}{2\nu} - A\right) e^{2\nu(y-h)} e^{A(y + h)} - \left(\frac{v_0}{2\nu} + A\right) e^{2\nu(y-h)} e^{-A(y + h)}\right) = C_1
\]

Second stream line

\[
\Rightarrow \frac{v_0 \text{Sinh} \ 2Ah}{U} x - \frac{Ke^{2\nu(y-h)}}{2} \left\{\frac{v_0}{2\nu} \left(e^{A(y + h)} - e^{-A(y + h)}\right) - A \left(e^{A(y + h)} + e^{-A(y + h)}\right)\right\} = C_1
\]

\[
\frac{v_0 \text{Sinh} \ 2Ah}{U} x - \frac{Ke^{2\nu(y-h)}}{2} \left\{\frac{v_0}{2\nu} \text{Sinh} \ A(y + h) - 2A \text{Cosh} \ A(y + h)\right\} = C_1
\]

First stream line

\[
\Rightarrow \frac{v_0 \text{Sinh} \ 2Ah}{U} x = - K e^{2\nu(y-h)} \left\{\frac{v_0}{2\nu} \text{Sinh} \ A(y + h) - A \text{Cosh} \ A(y + h)\right\} = C_1 \quad \ldots (14)
\]

Second stream line

\[
z = C_2 \quad \ldots (15)
\]
Now the curl $\vec{q} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\nu_0}{2\nu} \frac{\sinh A(y + h) + \cosh A(y + h)}{\sinh 2Ah} & v_0 & 0 \end{vmatrix}

= -\frac{U e^{2\nu(y - h)}}{\sinh 2Ah} \left[ \frac{\nu_0}{2\nu} \sinh A(y + h) + \cosh A(y + h) \right] \neq 0 \therefore \text{Motion of fluid is rotational}

Table for velocity: when the $y$ & $h$ are vary and other are fixed

\[
\text{let } U = 6, \quad \mu = .5, \quad K = \frac{1}{3}, \quad \frac{\nu_0}{2\nu} = 2, \quad \& \quad \sqrt{\left(\frac{\nu_0}{2\nu}\right)^2 - \frac{1}{K}} = \sqrt{4 - 3} = 1 = A
\]

<table>
<thead>
<tr>
<th>$h$</th>
<th>$y$</th>
<th>0</th>
<th>0.1</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
<th>0.6</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.3</td>
<td>u(y)</td>
<td>1.575</td>
<td>2.595</td>
<td>4.021</td>
<td>6</td>
<td>8.732</td>
<td>12.49</td>
<td>17.63</td>
</tr>
<tr>
<td>0.4</td>
<td>u(y)</td>
<td>1.245</td>
<td>1.93</td>
<td>2.88</td>
<td>4.196</td>
<td>6</td>
<td>8.47</td>
<td>11.84</td>
</tr>
<tr>
<td>0.5</td>
<td>u(y)</td>
<td>0.979</td>
<td>1.46</td>
<td>2.126</td>
<td>3.04</td>
<td>4.29</td>
<td>6</td>
<td>8.33</td>
</tr>
<tr>
<td>0.6</td>
<td>u(y)</td>
<td>.762</td>
<td>1.11</td>
<td>1.59</td>
<td>2.24</td>
<td>3.13</td>
<td>4.35</td>
<td>6</td>
</tr>
<tr>
<td>0.7</td>
<td>u(y)</td>
<td>0.589</td>
<td>0.843</td>
<td>1.19</td>
<td>1.66</td>
<td>2.31</td>
<td>3.19</td>
<td>4.38</td>
</tr>
</tbody>
</table>
Table for skin friction:

\[ \mu = 0.5, \quad \frac{v_0}{2u} = 2, \quad \frac{1}{K} = 4 - 3 = 1 = A \]

<table>
<thead>
<tr>
<th>h</th>
<th>y</th>
<th>0</th>
<th>0.1</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
<th>0.6</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.3</td>
<td>(\sigma_{xy})</td>
<td>4.28</td>
<td>6.01</td>
<td>8.37</td>
<td>11.59</td>
<td>15.96</td>
<td>21.89</td>
<td>29.93</td>
</tr>
<tr>
<td>0.4</td>
<td>(\sigma_{xy})</td>
<td>2.89</td>
<td>4.02</td>
<td>5.57</td>
<td>7.67</td>
<td>10.52</td>
<td>14.38</td>
<td>19.62</td>
</tr>
<tr>
<td>0.5</td>
<td>(\sigma_{xy})</td>
<td>2.04</td>
<td>2.82</td>
<td>3.88</td>
<td>5.32</td>
<td>7.29</td>
<td>9.94</td>
<td>13.53</td>
</tr>
<tr>
<td>0.6</td>
<td>(\sigma_{xy})</td>
<td>1.47</td>
<td>2.03</td>
<td>2.78</td>
<td>3.8</td>
<td>5.19</td>
<td>7.06</td>
<td>9.56</td>
</tr>
<tr>
<td>0.7</td>
<td>(\sigma_{xy})</td>
<td>1.08</td>
<td>1.48</td>
<td>2.02</td>
<td>2.76</td>
<td>3.75</td>
<td>5.10</td>
<td>6.92</td>
</tr>
</tbody>
</table>
Table for velocity: when y & A are vary and other are fixed

\[ U = 6, \quad \mu = .5, \quad \frac{V_0}{2\nu} = 6, h = .5, \quad \sqrt{\frac{V_0^2}{2\nu}} - \frac{1}{K} = A \]

Table 3: (for velocity)

<table>
<thead>
<tr>
<th>A</th>
<th>y</th>
<th>0</th>
<th>0.1</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
<th>0.6</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>u(y)</td>
<td>.132</td>
<td>.295</td>
<td>.64</td>
<td>1.367</td>
<td>2.88</td>
<td>6</td>
<td>12.42</td>
</tr>
<tr>
<td>2</td>
<td>u(y)</td>
<td>.092</td>
<td>.227</td>
<td>.52</td>
<td>1.184</td>
<td>2.67</td>
<td>6</td>
<td>13.44</td>
</tr>
<tr>
<td>3</td>
<td>u(y)</td>
<td>.063</td>
<td>.16</td>
<td>.398</td>
<td>.99</td>
<td>2.43</td>
<td>6</td>
<td>14.77</td>
</tr>
<tr>
<td>4</td>
<td>u(y)</td>
<td>.04</td>
<td>.11</td>
<td>.298</td>
<td>.811</td>
<td>2.21</td>
<td>6</td>
<td>16.31</td>
</tr>
<tr>
<td>5</td>
<td>u(y)</td>
<td>.024</td>
<td>.073</td>
<td>.221</td>
<td>.665</td>
<td>1.997</td>
<td>6</td>
<td>18.025</td>
</tr>
</tbody>
</table>
Table for skin friction: when $y$ & $A$ are vary and other are fixed

\[ U = 6, \quad \mu = .5, \quad \frac{v_0}{2U} = 6, \quad h = .5, \quad \sqrt{\left(\frac{v_0}{2U}\right)^2 - \frac{1}{K}} = A \]
4. CONCLUSION AND DISCUSSION

In this paper, we have investigated the velocity by the graphs of table-1 of equation (5) between velocity and distance in porous medium. The velocity increases in the interval $0 \leq y \leq 0.6$ at each value of $h$ lies between 0.3 to 0.7. Again value of velocity decreases correspondingly at each value of $y$ from 0 to 0.6 when $h$ increases.

Again from the table-3 the velocity increases in the interval $0 \leq y \leq 0.6$ at each value of $A$ lies between 1 to 5, velocity is equal ${u(y) = 6}$ at $y = 0.5$ at each value of $A$. But the velocity decreases correspondingly in the interval $0 \leq y \leq 0.4$ and the velocity increases at $y = 0.6$ at each value of $A$ lies from 1 to 5.
Similarly from the table-2 the value of skin friction increases in the interval \(0 \leq y \leq 0.6\) at each value of \(h\) lies from 0.3 to 0.7. Again skin friction decreases correspondingly at each value of \(y\) when \(h\) increases from .3 to .7.

Again from the table-4 the value of skin friction increases in the interval \(0 \leq y \leq 0.6\) at each value of \(\alpha\) increases 1 to 5. Again skin friction decreases correspondingly in the interval \(0 \leq y \leq 0.3\) and increases in the interval \(0.4 \leq y \leq 0.6\) when \(\alpha\) increases from 1 to 5. Also we have investigated shearing stress, the volumetric flow, drag coefficients and stream lines by the equations (7), (9), (11), (12), (13), (14) and (15) respectively.

REFERENCES


