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# Time-Optimal Control Problem for $n\times n$ Co-Operative Parabolic Systems with Strong Constraint Control in Initial Conditions

[1] **Byung Soo Lee,** [2] **Mohammed Shehata,** [3] **Salahuddin**

[1] Prof., Department of Mathematics, Kyungsung University, Korea

[2] Assist. Prof., Department of Mathematics, Faculty of Science, Jazan University, Kingdom of Saudi Arabia

[3] Assoc. Prof., Department of Mathematics, Faculty of Science, Jazan University, Kingdom of Saudi Arabia

[1] bslee@ks.ac.kr, [2] mashehata_math@yahoo.com, [3] salahuddin12@mailcity.com

## ABSTRACT

In this communication we considered and studied the time-optimal control problem for a liner $n\times n$ co-operative parabolic system defined on a bounded open domain $\Omega\subseteq \mathrm{R}^{N}$ with a strong constraint control $u\in U\subset (H_0^1(\Omega))^n$. This problem is, steering an initial state $y(0)=u$ with a control $u$, so that an observation $y(t)$ hitting a given target set in minimum time. First, we proved the existence and uniqueness of a solution of system under assumptions of the coefficients and also discussed the necessary and sufficient conditions of optimality.

**Keywords:** *Time-optimal control problems, bang-bang controls, parabolic system,* $n\times n$ *co-operative*

## 1. INTRODUCTION

The *time optimal* control problem is plays an important role in the field of control theory. The general version is that steering the initial state $y_0$ in a Hilbert space $H$ to hitting a target set $K\subset H$ in minimum time, with a *control* subject to constraints $(u\in U\subset H)$. In this communication our target is to high light some special aspects of minimum time problems for $n\times n$ co-operative parabolic system involving Laplace operators with control acts in the initial conditions.

Let $V$ and $H$ be two real Hilbert spaces and $V$ be a dense subspace of $H$. $H'$ is a dual of $H$ we may consider $V\subset H\subset V'$, where the embedding is dense. Let $A(t)$ ( $t\in ]0,T[$ ) be a family of continuous operators associated with a bilinear form $\pi(t;\cdot,\cdot)$ defined on $V\times V$ satisfying the following Gårding's inequality:

$$\pi(t;y,y)+c_0\|y\|_H^2\geq c_1\|y\|_V^2,\quad c_0\geq 0, c_1>0,\quad \text{for } y\in V, t\in[0,T]. \tag{1}$$

It is known, from [9] and [10] that for a bounded linear operator $B$ on $H$, the following abstract system;

$$\left.\begin{aligned}&\frac{d}{dt}y(t)+A(t)y(t)=f,\ f\in L^2(0,T;V'),\\&y(0)=Bu\end{aligned}\right\} \tag{2}$$

has a unique weak solution $y\in C([0,T];H)$ for $t\in]0,T[$ . We shall denote by $y(t;u)$ the unique solution of the system (2) corresponding to the control u. The time-optimal control problem we shall concern reads:

$$\min\{\tau: y(\tau;u)\in K, u\in U\}, \tag{3}$$

Where $K$ is a given subset of $H$, which is called the target set of the problem (3). A control $u^0$ is called a time-optimal control if $u^0\in U$ and there exists a number $\tau^0>0$ such that $y(\tau^0;u^0)\in K$ and

$$\tau^0=\min\{\tau: y(\tau;u)\in K, u\in U\} \tag{4}$$

where $\tau^0$ denotes the optimal time for the time-optimal control problem (4).

Three questions (problems) arise basically in connection with this problem.

a) Exist there a control $u$ and $\tau>0$ such that $y(\tau;u)\in K$? (this is an approximate controllability problem).
b) Assume that the answer to a) is in the affirmative and $\tau^0=\min\{\tau: y(\tau;u)\in K, u\in U\}$. Does there exist a control $u^0$ which steering $y(\tau^0)$ to hitting a target set $K$ in minimum time?
c) If $u^0$ exists, is it unique? what additional properties does it have?

Let $\Omega \subset \mathsf{R}^N$ be a bounded open domain with a smooth boundary $\Gamma$, and set $Q = \Omega \times ]0,T[$ and $\Sigma = \Gamma \times ]0,T[$. From [3] and [9], the existence of time optimal controls of the following controlled linear parabolic equations with the distributed control $u$ was obtained:

$$\left.\begin{aligned} &\frac{\partial y}{\partial t} = \Delta y + u && \text{in } Q, \\ &y(x,0) = y_0(x) && \text{in } \Omega, \\ &y(x,t) = 0 && \text{on } \Sigma, \end{aligned}\right\} \qquad (5)$$

Where $y_0(x)$ is a given function in $L^2(\Omega)$, $u \in U$ and $U$ is a closed bounded set in $L^2(\Omega)$. The results in [3] partly overlap with the results in [9] and they were shown that if the system (5) is controllable and if $K = \{0\}$ then the corresponding time-optimal control problem has at least one solution and it is bang-bang.

In [11], the authors gave a sufficient and necessary condition for the existence of the time-optimal control of the problem with the target set $K = \{0\}$ and certain controlled systems. Consider the following controlled system:

$$\left.\begin{aligned} &\frac{\partial y}{\partial t} = \Delta y + ay + u && \text{in } Q, \\ &y(x,0) = y_0(x) && \text{in } \Omega, \\ &y(x,t) = 0 && \text{on } \Sigma, \end{aligned}\right\} \qquad (6)$$

Where $a$ is a real number. Let $\{\lambda_i\}_{i\geq 1}, \lambda_1 \leq \lambda_2 \leq \lambda_3 ....,$ be the eigenvalues of $-\Delta$ with the Dirichlet boundary condition and $\{e_i\}_{i\geq 1}$ be the corresponding eigenfunctions, which is an orthogonal basis of $L^2(\Omega)$. We consider the target set $K$ to be the origin $\{0\}$ in $L^2(\Omega)$ and the control set $U$ to be the set

$$U_\varepsilon = \{u(\cdot,t) \in L^2(\Omega) : \|u\|_{L^2(\Omega)} \leq \varepsilon\}$$

Where $\varepsilon$ is a positive number, namely, $U_\varepsilon = B(0,\varepsilon)$, the closed ball in $L^2(\Omega)$ centered at $0$ and of radius $\varepsilon$. It was proved that if $K = \{0\}$ and $U = U_\varepsilon$, then the corresponding time-optimal control problem has at least one solution if and only if $a \leq \lambda_1$.

Very recently the time-optimal controls system for globally controlled linear and semilinear parabolic equations was studied by [4], [6] and [8]. Latter on the optimal control of an infinite order hyperbolic equation with a control via initial conditions was considered by [7].

In 2013, Shehata [13] considered the time-optimal control problem for $n \times n$ co-operative parabolic systems with a control $u \in L^2(\Omega)$ . Now we extend the Shehata [13] results to the case of strong constraint $u \in H_0^1(\Omega)$.

Inspired by [1, 2, 12,14], the time-optimal control problem of $n \times n$ co-operative hyperbolic systems with different cases of the observation and distributed or boundary controls constraints was considered and proved the existence and uniqueness of solutions for $n \times n$ co-operative parabolic system under conditions on the coefficients stated by the principal eigenvalue of the Laplace eigenvalue problem.

## 2. $n \times n$ CO-OPERATIVE PARABOLIC SYSTEMS IN SOBOLEV

Let $H_0^1(\Omega)$ be the usual Sobolev space of order one which consists of all $\phi \in L^2(\Omega)$ whose distributional derivatives $\dfrac{\partial \phi}{\partial x_i} \in L^2(\Omega)$ and $\phi_\Gamma = 0$ with the scalar product norm

$$< y,\phi >_{H_0^1(\Omega)} =< y,\phi >_{L^2(\Omega)} + < \nabla y, \nabla \phi >_{L^2(\Omega)},$$

$$\text{where } \nabla = \sum_{k=1}^{N} \frac{\partial}{\partial x_k}.$$

We have the following dense embedding chain

$$(H_0^1(\Omega))^n \subseteq (L^2(\Omega))^n \subseteq (H_0^{-1}(\Omega))^n,$$

where $H_0^{-1}(\Omega)$ is the dual of $H_0^1(\Omega)$.

Here and everywhere below the vectors are denoted by bold letters. For $\mathbf{y} = (y_i)_{i=1}^n, \boldsymbol{\phi} = (\phi_i)_{i=1}^n \in (H_0^1(\Omega))^n$ and $t \in ]0,T[$ , let us define a family of continues bilinear forms $\pi(t;\cdot,\cdot): (H_0^1(\Omega))^n \times (H_0^1(\Omega))^n \to \mathsf{R}$ by

$$\pi(t;\mathbf{y},\phi) \begin{array}{l} = \sum_{i=1}^{n}\int_{\Omega}[(\nabla y_i)(\nabla \phi_i) - a_i(x,t)y_i\phi_i]dx \\ - \sum_{i,j=1}^{n}\int_{\Omega}a_{ij}(x,t)y_j\phi_i dx, \end{array} \quad (7)$$

where

$$\left.\begin{array}{l} a_i(x,t) \text{ and } a_{ij}(x,t) \text{ are positive} \\ \text{functions in } L^{\infty}(Q), \\ a_{ij} = 0 \quad \text{when } i = j \quad \text{and} \\ a_{ij} \le \sqrt{a_i a_j} \quad \text{when } i \ne j \end{array}\right\} \quad (8)$$

The bilinear form (7) can be put in the operator form:

$$\pi(t;\mathbf{y},\phi) \begin{array}{l} = \sum_{i=1}^{n}\int_{\Omega}[(-\Delta y_i) - a_i(x,t)y_i]\phi_i dx \\ - \sum_{i,j=1}^{n}\int_{\Omega}a_{ij}(x,t)y_j\phi_i dx \\ = \sum_{i=1}^{n} < -(A(t)\mathbf{y})_i, \phi >_{L^2(\Omega)}, \end{array}$$

Where $A(t)$ is in $n \times n$ matrix operator which maps $(H_0^1(\Omega))^n$ onto $(H_0^{-1}(\Omega))^n$ and takes the form

$$A(t)\mathbf{y} = \begin{pmatrix} \Delta + a_1 & a_{12} & & a_{1n} \\ a_{21} & \Delta + a_2 & & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & & \Delta + a_n \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$$

.

**Lemma 1:**

[5] If $\Omega$ is a regular bounded domain in $\mathsf{R}^{\mathsf{N}}$, with boundary $\Gamma$, and if $m$ is positive on $\Omega$ and smooth enough (in particular $m \in L^{\infty}(\Omega)$), then the eigenvalue problem:

$$\left.\begin{array}{ll} -\Delta y = \lambda m(x) y & \text{in } \Omega, \\ y = 0 & \text{on } \Gamma \end{array}\right\}$$

possesses an infinite sequence of positive eigenvalues:

$$0 < \lambda_1(m) < \lambda_2(m) \le \cdots \lambda_k(m) \cdots ; \lambda_k(m) \to \infty, \text{ as } k \to \infty.$$

Moreover $\lambda_1(m)$ is simple, its associate eigenfunction $e_m$ is positive, and $\lambda_1(m)$ is characterized by

$$\lambda_1(m)\int_{\Omega} m y^2 dx \le \int_{\Omega}|\nabla y|^2 dx \quad (9)$$

Now, let

$$\lambda_1(a_i) \ge n, \quad i = 1,2,\ldots\ldots,n \quad (10)$$

**Lemma 2:**

If (8) and (10) are hold, then (7) satisfies the Gårding inequality

$$\pi(t;\mathbf{y},\mathbf{y}) + c_0\|y\|^2_{\left(L^2(\Omega)\right)^n} \ge c_1\|y\|^2_{\left(H_0^1(\Omega)\right)^n}, \quad c_0, c_1 > 0$$

Proof. In fact

$$\pi(t;\mathbf{y},\mathbf{y}) \begin{array}{l} = \sum_{i=1}^{n}\int_{\Omega}\left[|\nabla y_i|^2 - a_i(x,t)y_i^2\right]dx \\ - \sum_{i,j=1}^{n}\int_{\Omega}a_{ij}(x,t)y_i y_j dx \\ \ge \sum_{i=1}^{n}\int_{\Omega}\left[|\nabla y_i|^2 - a_i(x,t)y_i^2\right]dx \\ - 2\sum_{i>j}^{n}\int_{\Omega}\sqrt{a_i(x,t)a_j(x,t)}\,y_i y_j dx \end{array}$$

By using the Cauchy Schwarz inequality and (9), we get

$$\pi(t;\mathbf{y},\mathbf{y}) \ge \begin{array}{l} \sum_{i=1}^{n}\left(1 - \frac{1}{\lambda_1(a_i)}\right)\int_{\Omega}|\nabla y_i|^2 dx \\ - 2\sum_{i>j}^{n}\int_{\Omega}\left(\frac{1}{\sqrt{\lambda_1(a_i)\lambda_1(a_j)}}\right) \\ \left(\int_{\Omega}|\nabla y_i|^2 dx\right)^{\frac{1}{2}}\left(\int_{\Omega}|\nabla y_j|^2 dx\right)^{\frac{1}{2}} \\ \ge \sum_{i=1}^{n}\left(\frac{\lambda_1(a_i) - n}{\lambda_1(a_i)}\right)\int_{\Omega}|\nabla y_i|^2 dx \end{array}$$

From (10), we have

$$\pi(t;\mathbf{y},\mathbf{y}) \ge \alpha\left[\sum_{i=1}^{n}\int_{\Omega}|\nabla y_i|^2 dx\right] \quad \alpha > 0$$

Adding $\|\mathbf{y}\|_{(L^2(\Omega))^n}$ to two sides, then we get the desired result.

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We can now apply the results of [10, p.33] to obtain the following theorem:

**Theorem 1:**

If (8) and (10) hold, then there exist a unique solution

$$\mathbf{y} \in H^{2,1}(\Omega) = L^2\left(0,T;H_0^1(\Omega)\right) \cap H^1(0,T;L^2(\Omega))$$

satisfying the following $n \times n$ system: $i = 1,2,...,n$

$$\left.\begin{aligned} &\frac{\partial y_i}{\partial t} = \left(A(t)\mathbf{y}\right)_i + f_i, \quad f_i \in L^2(Q) && \text{in } Q,\\ &y_i(x,0) = u_i(x), \quad u_i(x) \in H_0^1(\Omega) && \text{in } \Omega,\\ &y_i(x,t) = 0 && \text{on } \Sigma. \end{aligned}\right\} \quad (11)$$

Moreover, $\mathbf{y}$ is continuous from $[0,T]$ to $(H_0^1(\Omega))^n$

## 3. MINIMUM TIME AND CONTROLLABILITY IN SOBOLEV SPACE

We denote the unique solution of (11), at time $t$ for each control $\mathbf{u} = (u_1, u_2, ..., u_n)$ by $\mathbf{y}(t;\mathbf{u})$ and we write $\mathbf{y}(x,t;\mathbf{u})$ when the explicit dependence on $x$ if necessary. Now we define the time-optimal control problem corresponding to the $n \times n$ co-operative parabolic system (11):

$$\min\{t : \mathbf{y}(t;\mathbf{u}) \in K_\varepsilon^n, \mathbf{u} \in U_\varepsilon^n\}, \qquad (12)$$

with constraints

$$\left.\begin{aligned} &\mathbf{y}(t;\mathbf{u}) \text{ is the solution of (11)},\\ &U_\varepsilon^n = \{\mathbf{u} = (u_1,u_2,...,u_n) \in (H_0^1(\Omega))^n :\\ &\|u_i\|_{H_0^1(\Omega)} \le \varepsilon\},\\ &K_\varepsilon^n = \{\mathbf{z} = (z_1,z_2,...,z_n) \in (L^2(\Omega))^n :\\ &\|z_i - z_{id}\|_{L^2(\Omega)} + \sum_{j=1}^{N}\left\|\frac{\partial z_i}{\partial x_j} - z_{id}\right\|_{L^2(\Omega)} \le \varepsilon\}, \end{aligned}\right\} \quad (13)$$

and $\varepsilon, \varepsilon > 0$ and $z_{id} \in L^2(\Omega)$ are given.

**Theorem 2:**

If (8) and (10) are hold then the system (11) is controllable,

$$\text{i.e.,}\quad \text{there exists a } \tau \in ]0,T] \text{ and } \mathbf{u} \in U_\varepsilon^n \text{ with } \mathbf{y}(\tau;\mathbf{u}) \in K_\varepsilon^n \qquad (14)$$

**Proof:**

We can reduce the problem of controllability to the case of the system (11) with $f_i = 0$. Here $\mathbf{y}(\tau;\mathbf{u}) \in (H_0^1(\Omega))^n$. To show the system is controllable let $\psi_i(x) \in H^{-1}(\Omega)$ such that

$$< \psi_i(x), y_i(x,\tau;\mathbf{u}) >= 0 \quad \forall \mathbf{u} \in (H_0^1(\Omega))^n,$$

where $<\cdot,\cdot>$ denotes the duality paring between

$$H^{-1}(\Omega) \text{ and } H_0^1(\Omega).$$

Let us introduce the adjoint state $p(t;\mathbf{u})$ by the solution of the following system

$$\left.\begin{aligned} &-\frac{\partial p_i}{\partial t}(t;\mathbf{u}) - \left(A^*(t)\mathbf{p}(t;\mathbf{u})\right)_i = 0 \quad \text{in}\,\Omega \times ]0,\tau[,\\ &p_i(x,\tau) = \bar{z}_i(x) \in H^{-1}(\Omega) \quad \text{in}\,\Omega,\\ &p_i(x,t) = 0 \quad \text{in}\,\Gamma \times ]0,\tau[, \end{aligned}\right\} (15)$$

where $A^*(t)$ is the adjoint of $A(t)$ which is defined by

$$< A^*(t)\phi, \psi >=< \phi, A(t)\psi >, \quad \phi, \psi \in (H_0^1(\Omega))^n.$$

Since $\bar{z}_i \in H^{-1}(\Omega)$, the existence of a unique weak solution $\mathbf{p}$ for (15) can be proved by the transposition (see Chapter 3 in [9]). Multiplying the first equation in (15) by $y_i(t;\mathbf{u})$ and integrate by parts from $0$ to $\tau$, we obtain the following identity;

$$\int_0^\tau \int_\Omega p_i(x,t;\mathbf{u})u_i\,dxdt =< \bar{z}_i(x), y_i(x,\tau;\mathbf{u}) >= 0.$$

But from the continuity property, $p_i(\tau;\mathbf{u}) \equiv 0$ and hence $\bar{z}_i = 0$.

Now set

$$\tau^0 = \inf\{\tau : \mathbf{y}(\tau;\mathbf{u}) \in K_\varepsilon^n \text{ for some } \mathbf{u} \in U_\varepsilon^n\}. \qquad (16)$$

Then, the following result holds.

**Theorem 3:**

If (8) and (10) are hold, then there exists an admissible control $\mathbf{u}^0$ to the problem (12)-(16), which steering $\mathbf{y}(t;\mathbf{u}^0)$ to hit a target set $K_\varepsilon^n$ in minimum time $\tau^0$ (defined by (16)). Moreover

$$\sum_{i=1}^{n}\int_\Omega (I-\Delta)(y_i(\tau^0;\mathbf{u}^0)-z_{id}) \left(y_i(\tau^0;\mathbf{u})-y_i(\tau^0;\mathbf{u}^0)\right)dx \geq 0 \quad \forall \mathbf{u}\in U_\varepsilon^n \tag{17}$$

**Proof:**

Fixe $x$, we can choose $\tau^m \to \tau^0$ and admissible controls $\{\mathbf{u}^m\}$ such that

$$\mathbf{y}(\tau^m;\mathbf{u}^m)\in K_\varepsilon^n,\quad m=1,2,\ldots.$$

Set $\mathbf{y}^m=\mathbf{y}(\mathbf{u}^m)$. Since $U_\varepsilon^n$ is bounded, we may verify that $\mathbf{y}^m$ ranges in a bounded set in

$$(L^2(0,T;(L^2(\Omega))^n)=(L^2(Q))^n).$$

Again We may extract a subsequence denoted by $\{\mathbf{u}^m,\mathbf{y}^m\}$ such that

$$\left.\begin{array}{l}\mathbf{u}^m \to \mathbf{u}^0 \quad \text{weakly in } (H_0^1(\Omega))^n,\quad u^0\in U_\varepsilon^n,\\ \mathbf{y}^m\to \mathbf{y}\quad \text{weakly in } L^2\left(0,T;\left(H_0^1(\Omega)\right)^n\right)\end{array}\right\} \tag{18}$$

We deduce from the equality

$$\frac{d\mathbf{y}^m}{dt}=f-A(t)\mathbf{y}^m$$

that is

$$\frac{d\mathbf{y}^m}{dt}\to\frac{d\mathbf{y}}{dt}=f-A(t)\mathbf{y}\quad \text{in} L^2\left(0,T;\left(H^{-1}(\Omega)\right)^n\right),$$

And

$$\mathbf{y}^m(0)\to\mathbf{y}(0)=\mathbf{u}^0\quad \text{in} U_\varepsilon^n.$$

But

$$\mathbf{y}(\tau^m;\mathbf{u}^m)-\mathbf{y}(\tau^0;\mathbf{u}^0)\begin{array}{l}=\mathbf{y}(\tau^m;\mathbf{u}^m)-\mathbf{y}(\tau^0;\mathbf{u}^m)\\ +\mathbf{y}(\tau^0;\mathbf{u}^m)-\mathbf{y}(\tau^0;\mathbf{u}^0)\end{array}$$

Now from (18)

$$\mathbf{y}(\tau^0;\mathbf{u}^m)\to\mathbf{y}(\tau^0;\mathbf{u}^0)\quad \text{weakly in} (H_0^1(\Omega))^n \tag{19}$$

and

$$\begin{aligned}&\left\|\mathbf{y}(\tau^m;\mathbf{u}^m)-\mathbf{y}(\tau^0;\mathbf{u}^m)\right\|_{\left(H^{-1}(\Omega)\right)^n}\\ &=\left\|\int_{\tau^0}^{\tau^m}\frac{d}{dt}\mathbf{y}(t;\mathbf{u}^m)dt\right\|_{\left(H^{-1}(\Omega)\right)^n}\\ &\leq\sqrt{\tau^m-\tau^0}\left(\int_{\tau^0}^{\tau^m}\left\|\frac{d}{dt}\mathbf{y}(t;\mathbf{u}^m)dt\right\|^2_{\left(H^{-1}(\Omega)\right)^n}dt\right)^{\frac{1}{2}}\\ &\leq c\sqrt{\tau_n-\tau^0}\end{aligned} \tag{20}$$

Combining (19) and (20), we get

$$\mathbf{y}(\tau^m;\mathbf{u}^m)-\mathbf{y}(\tau^0;\mathbf{u}^0)\to 0\quad \text{weakly in} (H_0^{-1}(\Omega))^n. \tag{21}$$

Since $\mathbf{y}(\tau^0;\mathbf{u}^0)\in K_\varepsilon^n$ as $K_\varepsilon^n$ is closed and convex, hence weakly closed. This shows that $K_\varepsilon^n$ is reached in time $\tau^0$ by the admissible control $\mathbf{u}^0$ .

For the second part of the theorem, really, from Theorem 1, the mapping $t\to\mathbf{y}(t;\mathbf{u})$ from $[0,T]\to(H_0^1(\Omega))^n$ is continuous for each fixed $\mathbf{u}$ and so $\mathbf{y}(\tau^0;\mathbf{u})\notin intK_\varepsilon^n$, for any $\mathbf{u}\in U_\varepsilon^n$, by the minimality of $\tau^0$.

Using Theorem 1 it is easy to verify that the mapping $\mathbf{u}\to\mathbf{y}(\tau^0;\mathbf{u})$ , defined on $(H_0^1(\Omega))^n$ , is continuous and linear. then, the set

$$\mathsf{A}(\tau^0)=\{\mathbf{y}(\tau^0;\mathbf{u}):\mathbf{u}\in U_\varepsilon^n\}$$

is the image under a linear mapping of a convex set, hence $\mathsf{A}(\tau^0)$ is convex. Thus we have

$$\mathsf{A}(\tau^0)\cap intK_\varepsilon^n=\varnothing$$

and $\mathbf{y}(\tau^0;\mathbf{u}^0)\in\partial K_\varepsilon^n$ (the boundary of $K_\varepsilon^n$ ). Since $intK_\varepsilon^n\neq\varnothing$ *from* (14) there exists a closed hyperplane separating $A(\tau^0)$ and $K_\varepsilon^n$ containing $\mathbf{y}(\tau^0;\mathbf{u}^0)$ , i.e., there is a nonzero $\mathbf{g}\in(L^2(\Omega))^n$ such as

$$<\mathbf{g},\mathbf{y}(\tau^0;\mathbf{u})>\leq<\mathbf{g},\mathbf{y}(\tau^0;\mathbf{u}^0)>\leq\inf_{\mathrm{y}\in K_\varepsilon^n}<\mathbf{g},\mathbf{y}(\tau^0;\mathbf{u})> \tag{22}$$

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From the second inequality in (22), $\mathbf{g}$ must support the set $K_\varepsilon^n$ at $\mathbf{y}(\tau^0;\mathbf{u}^0)$ i.e.,

$$<\mathbf{g},(\mathbf{y}(\tau^0;\mathbf{u})-\mathbf{y}(\tau^0;\mathbf{y}^0))>\geq 0 \quad \forall \mathbf{u}\in U_\varepsilon^n$$

and since $(H_0^1(\Omega))^n$ is a Hilbert space, $\mathbf{g}$ must be of the form

$$\mathbf{g}=\lambda(\mathbf{y}(\tau^0;\mathbf{u}^0)-z_{id}) \quad \text{for some } \lambda>0.$$

Dividing the inequality (22) by $\lambda$ gives the desired result.

Now the inequality (17) can be interpreted as follows: let us introduce the adjoint state $p(t;\mathbf{u}^0)$ by the solution of the following system

$$\left.\begin{aligned}
&-\frac{\partial p_i}{\partial t}(t;\mathbf{u}^0)+\left(A^*(t)\mathbf{p}(t;\mathbf{u}^0)\right)_i=0 \quad \text{in}\,\Omega\times]0,\tau^0[,\\
&p_i(x,\tau^0)=(I-\Delta)(y_i(x,\tau^0)-z_{id}) \quad \text{in}\,\Omega,\\
&p_i(x,t)=0 \quad \text{in}\,\Gamma\times]0,\tau^0[,
\end{aligned}\right\} \tag{23}$$

As the proof of Theorem 2, we multiply the first equation int (23) by $y_i(t;\mathbf{u})-y_i(t;\mathbf{u}^0)$ and integrate by parts from $0$ to $\tau^0$, we obtain the following identity:

$$\begin{aligned}
&\int_\Omega (I-\Delta)(y_i(\tau^0;\mathbf{u}^0)-z_{id})\\
&(y_i(x,\tau^0;\mathbf{u})-y_i(x,\tau^0;\mathbf{u}^0))dx\\
&=\int_\Omega p_i(0;\mathbf{u}^0)(\mathbf{u}-\mathbf{u}^0)dx.
\end{aligned}$$

Hence the inequality (17) becomes

$$\sum_{i=1}^n\int_\Omega p_i(x,0;\mathbf{u}^0)(\mathbf{u}-\mathbf{u}^0)dx\geq 0 \quad \forall \mathbf{u}\in U_\varepsilon^n. \tag{24}$$

Using the controllability condition (14), we have that the backward uniqueness property implies $p_i(x,0;\mathbf{u}^0)=0$. Hence the optimal control is bang-bang, i.e., $\|u_i^0\|_{L^2(\Omega)}=\varepsilon$ and since $U_\varepsilon^n$ is strictly convex, the optimal control is unique. We have thus proved:

**Theorem 4:**

If (8) and (10) are hold, then there exists the adjoint state $\mathbf{p}\in L^2(0,\tau^0;(L^2(\Omega))^n)$ such that the optimal control $\mathbf{u^0}$ of problem (12)-(16) is bang-bang and unique, which is determined by (23) and (24) together with (11) (with $u_i=u_i^0, i=1,2,...,n$ ).

## 4. SCALAR CASE

Here, we take the case where $n=2$, the time-optimal problem is

$$\min\{t: y(x,t;\mathbf{u})\in K_\varepsilon^2, \mathbf{u}=(u_1,u_2)\in U_\varepsilon^2\}.$$

The state $\mathbf{y}=(y_1,y_2)$ is solution of the following equations

$$\left.\begin{aligned}
&\frac{\partial y_1}{\partial t}-\Delta y_1=a_{11}(x,t)y_1+a_{12}(x,t)y_2+f_1,\\
&x\in\Omega,\quad t\in]0,\tau^0[,\\
&\frac{\partial y_2}{\partial t}-\Delta y_2=a_{21}(x,t)y_1+a_{22}(x,t)y_2+f_2,\\
&x\in\Omega,\quad t\in]0,\tau^0[,\\
&y_1(x,0)=u_1^0(x),\quad y_2(x,0)=u_2^0(x),\\
&x\in\Omega,\\
&y_1(x,t)=y_2(x,t)=0,\\
&x\in\Gamma,\quad t\in]0,\tau^0[,
\end{aligned}\right\}$$

with

$$\left.\begin{aligned}
&a_{ij}(x,t), i,j=1,2 \text{ are positive functions in } L^\infty(Q),\\
&\lambda_1(a_{11})\geq 2,\quad \lambda_1(a_{22})\geq 2.
\end{aligned}\right\}$$

The adjoint is a solution of the following equations

$$\left.\begin{aligned}
&-\frac{\partial p_1}{\partial t}-\Delta p_1=a_{22}(x,t)p_1-a_{12}(x,t)p_2+f_1,\\
&x\in\Omega,\quad t\in]0,\tau^0[\\
&-\frac{\partial p_2}{\partial t}-\Delta p_2=-a_{21}(x,t)p_1+a_{11}(x,t)p_2+f_2,\\
&x\in\Omega,\quad t\in]0,\tau^0[,\\
&p_1(x,\tau^0)=(I-\Delta)(y_1(x,\tau^0)-z_{1d}),\\
&x\in\Omega,\\
&p_2(x,\tau^0)=(I-\Delta)(y_2(x,\tau^0)-z_{2d}),\\
&x\in\Omega,\\
&p_1(x,t)=p_2(x,t)=0,\\
&x\in\Gamma,\quad t\in]0,\tau^0[.
\end{aligned}\right\}$$

The maximum condition is

$$\int_0^{\tau^0}\int_\Omega \begin{bmatrix} p_1(x,0;u^0)(u_1-u_1{}^0) \\ +\, p_1(x,0;u^0)(u_1-u_1{}^0) \end{bmatrix} dxdt \geq 0 \quad \forall u \in U_\varepsilon^2 .$$

## 5. COMMENTS

- In this communication we remark that if we have chosen to treat a special systems involving Laplace operator, most of the results we described without any change on the results, to more general parabolic systems involving the following second order operators:

$$L(x,.) = \sum_{i,j=1}^{n} b_{ij}(x,.)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{j=1}^{n} b_j(x,.)\frac{\partial}{\partial x_j} + b_0(x,.)$$

with sufficiently smooth coefficients (in particular, $b_{ij}, b_j, b_0 \in L^\infty(Q), b_j, b_0 > 0$ ) and under the

Legendre-Hadamard ellipticity condition

$$\sum_{i,j=1}^{n} \eta_i \eta_j \geq \sigma \sum_{i=1}^{n} \eta_i \quad \forall (x,t) \in Q,$$

for all $\eta_i \in \mathrm{R}$ and some constant $\sigma > 0$.

In this case we replace the first eigenvalue of the Laplace operator by the first eigenvalue of the operator $L$ (see [5]).

- If we have chosen to treat a co-operative parabolic systems with Dirichlet boundary conditions.The results can be extended to the case of $n \times n$ co-operative parabolic system with Neumann boundary conditions: If we take $H^1(\Omega)$ instead of $H_0^1(\Omega)$, we have to replace the Dirichlet boundary conditions $y_i = 0, p_i = 0$ on the boundary by Neumann boundary conditions $\frac{\partial y_i}{\partial \nu} = 0, \frac{\partial p_i}{\partial \nu} = 0$ where $\nu$ is the outward normal.

- The results in this paper, carry over to the fixed-time problem (Chapter 3 in [9])

$$\text{minimize} \sum_{i=1}^{n} \int_\Omega |\, y_i(x,T;\mathbf{u}) - z_{id}(x)|^2 \, dx, \quad T \text{ fixed},$$

subject to (11) (except the trivial case, where $z_{id}(x) = y_i(x,T;\mathbf{u}) \forall i = 1,2,..,n$ for some admissible control $\mathbf{u}$ ). This can be proven in an analogous manner, as the necessary and sufficient conditions for the optimality of this problem coincide with (11), (15) and (24) (with $u_i = u_i^0, i = 1,2,...,n$ ).

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