Time-Optimal Control Problem for $n \times n$ Co-Operative Parabolic Systems with Strong Constraint Control in Initial Conditions

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ABSTRACT

In this communication we considered and studied the time-optimal control problem for a linear $n \times n$ co-operative parabolic system defined on a bounded open domain $\Omega \subseteq \mathbb{R}^n$ with a strong constraint control $u \in U \subset (H^1_0(\Omega))^n$. This problem is, steering an initial state $y(0) = u$ with a control $u$, so that an observation $y(t)$ hitting a given target set in minimum time. First, we proved the existence and uniqueness of a solution of system under assumptions of the coefficients and also discussed the necessary and sufficient conditions of optimality.

Keywords: Time-optimal control problems, bang-bang controls, parabolic system, $n \times n$ co-operative

1. INTRODUCTION

The timeoptimal control problem is plays an important role in the field of control theory. The general version is that steering the initial state $y_0$ in a Hilbert space $H$ to hitting a target set $K \subset H$ in minimum time, with a control $u \in U \subset H$. In this communication our target is to highlight some special aspects of minimum time problems for $n \times n$ co-operative parabolic system involving Laplace operators with control acts in the initial conditions.

Let $V$ and $H$ be two real Hilbert spaces and $V$ be a dense subspace of $H$. $H'$ is a dual of $H$ we may consider $V \subset H \subset V'$, where the embedding is dense. Let $A(t) \ (t \in [0,T])$ be a family of continuous operators associated with a bilinear form $\pi(t,\cdot,\cdot)$ defined on $V \times V$ satisfying the following Gårding’s inequality:

$$\pi(t; y, y) + c_0 \|y\|^2 \geq c_1 \|y\|^2, \quad c_0 \geq 0, c_1 > 0,$$

for $y \in V$, $t \in [0,T]$. (1)

It is known, from [9] and [10] that for a bounded linear operator $B$ on $H$, the following abstract system;

$$\begin{aligned}
\frac{dy(t)}{dt} + A(t)y(t) &= f, f \in L^2(0,T;V'), \\
y(0) &= Bu
\end{aligned}$$

(2)

has a unique solution $y \in C([0,T];H)$ for $t \in [0,T]$. We shall denote by $y(t;u)$ the unique solution of the system (2) corresponding to the control $u$. The time-optimal control problem we shall concern reads:

$$\min \{\tau : y(\tau;u) \in K, u \in U\},$$

(3)

where $K$ is a given subset of $H$, which is called the target set of the problem (3). A control $u^0$ is called a time-optimal control if $u^0 \in U$ and there exists a number $\tau^0 > 0$ such that $y(\tau^0;u^0) \in K$ and

$$\tau^0 = \min \{\tau : y(\tau;u) \in K, u \in U\}$$

(4)

where $\tau^0$ denotes the optimal time for the time-optimal control problem (4).

Three questions (problems) arise basically in connection with this problem.

a) Exist there a control $u$ and $\tau > 0$ such that $y(\tau;u) \in K$? (this is an approximate controllability problem).

b) Assume that the answer to a) is in the affirmative and

$$\tau^0 = \min \{\tau : y(\tau;u) \in K, u \in U\}.$$

Does there exist a control $u^0$ which steering $y(\tau^0)$ to hitting a target set $K$ in minimum time?

c) If $u^0$ exists, is it unique? what additional properties does it have?
Let \( \Omega \subset \mathbb{R}^N \) be a bounded open domain with a smooth boundary \( \Gamma \), and set \( Q = \Omega \times ]0,T[ \) and \( \Sigma = \Gamma \times ]0,T[ \). From [3] and [9], the existence of time optimal controls of the following controlled linear parabolic equations with the distributed control \( u \) was obtained:

\[
\begin{align*}
\frac{\partial y}{\partial t} &= \Delta y + u \quad \text{in } Q, \\
y(x,0) &= y_0(x) \quad \text{in } \Omega, \\
y(x,t) &= 0 \quad \text{on } \Sigma,
\end{align*}
\]  

(5)

Where \( y_0(x) \) is a given function in \( L^2(\Omega) \), \( u \in U \) and \( U \) is a closed bounded set in \( L^2(\Omega) \). The results in [3] partly overlap with the results in [9] and they were shown that if the system (5) is controllable and if \( K = \{0\} \) then the corresponding time-optimal control problem has at least one solution and it is bang-bang.

In [11], the authors gave a sufficient and necessary condition for the existence of the time-optimal control of the problem with the target set \( K = \{0\} \) and certain controlled systems. Consider the following controlled system:

\[
\begin{align*}
\frac{\partial y}{\partial t} &= \Delta y + ay + u \quad \text{in } Q, \\
y(x,0) &= y_0(x) \quad \text{in } \Omega, \\
y(x,t) &= 0 \quad \text{on } \Sigma,
\end{align*}
\]  

(6)

Where \( a \) is a real number. Let \( \{\lambda_i\}_{i=1}^{\infty} \), \( \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots \), be the eigenvalues of \( -\Delta \) with the Dirichlet boundary condition and \( \{e_i\}_{i=1}^{\infty} \) be the corresponding eigenfunctions, which is an orthogonal basis of \( L^2(\Omega) \). We consider the target set \( K \) to be the origin \( \{0\} \) in \( L^2(\Omega) \) and the control set \( U \) to be the set

\[ U_\varepsilon = \{ u(\cdot, t) \in L^2(\Omega) : \| u \|_{L^2(\Omega)} \leq \varepsilon \} \]

Where \( \varepsilon \) is a positive number, namely, \( U_\varepsilon = B(0,\varepsilon) \), the closed ball in \( L^2(\Omega) \) centered at 0 and of radius \( \varepsilon \). It was proved that if \( K = \{0\} \) and \( U = U_\varepsilon \), then the corresponding time-optimal control problem has at least one solution if and only if \( a \leq \lambda_1 \).

Very recently the time-optimal controls system for globally controlled linear and semilinear parabolic equations was studied by [4], [6] and [8]. Latter on the optimal control of an infinite order hyperbolic equation with a control via initial conditions was considered by [7].

In 2013, Shehata [13] considered the time-optimal control problem for \( n \times n \) co-operative linear parabolic systems with a control \( u \in L^2(\Omega) \). Now we extend the Shehata [13] results to the case of strong constraint \( u \in H^1_0(\Omega) \).

Inspired by [1, 2, 12, 14], the time-optimal control problem of \( n \times n \) co-operative hyperbolic systems with different cases of the observation and distributed or boundary controls constraints was considered and proved the existence and uniqueness of solutions for \( n \times n \) co-operative parabolic system under conditions on the coefficients stated by the principal eigenvalue of the Laplace eigenvalue problem.

2. \( n \times n \) CO-OPERATIVE PARABOLIC SYSTEMS IN SOBOLEV

Let \( H^1_0(\Omega) \) be the usual Sobolev space of order one which consists of all \( \phi \in L^2(\Omega) \) whose distributional derivatives \( \frac{\partial \phi}{\partial x_i} \in L^2(\Omega) \) and \( \phi_t = 0 \) with the scalar product norm

\[ < y, \phi >_{H^1_0(\Omega)} = < y, \phi >_{L^2(\Omega)} + < \nabla y, \nabla \phi >_{L^2(\Omega)}, \]

where \( \nabla = \sum_{k=1}^{N} \frac{\partial}{\partial x_k} \).

We have the following dense embedding chain

\[ (H^1_0(\Omega))^n \subseteq (L^2(\Omega))^n \subseteq (H^1_0(\Omega))^n, \]

where \( H^1_0(\Omega) \) is the dual of \( H^1_0(\Omega) \).

Here and everywhere below the vectors are denoted by bold letters. For \( y = (y_{i_1})_{i_1=1}^{n}, \phi = (\phi_{i_1})_{i_1=1}^{n} \in (H^1_0(\Omega))^n \) and \( t \in ]0,T[ \), let us define a family of continuous bilinear forms\( \pi(t,\cdot): (H^1_0(\Omega))^n \times (H^1_0(\Omega))^n \rightarrow \mathbb{R} \) by
\[
\pi(t; y, \phi) = \sum_{i,j=1}^{n} \left[ (\nabla y_i)(\nabla \phi_j) - a_i(x,t)y_i \phi_j \right] dx \\
- \sum_{i,j=1}^{n} a_{ij}(x,t) y_i \phi_j dx,
\]

(7)

where \( a_{ij}(x,t) \) and \( a_{ji}(x,t) \) are positive functions in \( L^\infty(Q) \), \( a_{ii} = 0 \) when \( i = j \) and

\[
a_{ij} \leq \sqrt{a_{ii}a_{jj}} \quad \text{when} \quad i \neq j
\]

The bilinear form (7) can be put in the operator form:

\[
\pi(t; y, \phi) = \sum_{i,j=1}^{n} \left[ (\nabla y_i)(\nabla \phi_j) - a_i(x,t)y_i \phi_j \right] dx \\
- \sum_{i,j=1}^{n} a_{ij}(x,t) y_i \phi_j dx,
\]

(8)

Where \( A(t) \) is in \( n \times n \) matrix operator which maps \( (H^1_0(\Omega))^n \) onto \( (H^{-1}(\Omega))^n \) and takes the form

\[
A(t)y = \begin{pmatrix} 
\Delta + a_{11} & a_{12} & \cdots & a_{1n} \\
 a_{21} & \Delta + a_{22} & \cdots & a_{2n} \\
 \vdots & \vdots & \ddots & \vdots \\
 a_{n1} & a_{n2} & \cdots & \Delta + a_{nn}
\end{pmatrix} \begin{pmatrix} y_1 \\
y_2 \\
\vdots \\
y_n
\end{pmatrix}
\]

Lemma 1:

[5] If \( \Omega \) is a regular bounded domain in \( \mathbb{R}^N \), with boundary \( \Gamma \), and if \( m \) is positive on \( \Omega \) and smooth enough (in particular \( m \in L^\infty(\Omega) \)), then the eigenvalue problem:

\[
-\Delta y = \lambda m(x)y \quad \text{in} \quad \Omega, \\
y = 0 \quad \text{on} \quad \Gamma
\]

possesses an infinite sequence of positive eigenvalues:

\[
0 < \lambda_1(m) < \lambda_2(m) \leq \cdots \lambda_k(m) \cdots ; \lambda_k(m) \to \infty, \quad k \to \infty.
\]

Moreover \( \lambda_i(m) \) is simple, its associate eigenfunction \( e_m \) is positive, and \( \lambda_i(m) \) is characterized by

\[
\lambda_i(m) \int_{\Omega} m y_i^2 dx \leq \int_{\Omega} |\nabla y_i|^2 \ dx
\]

(9)

Now, let

\[
\lambda_i(a_i) \geq n, \quad i = 1, 2, \ldots, n
\]

(10)

Lemma 2:

If (8) and (10) are hold, then (7) satisfies the Gårding inequality

\[
\pi(t; y, y) + c_0 \left\| \nabla y \right\|^2_{L^2(\Omega)^n} \geq c_1 \left\| y \right\|_{H_0^1(\Omega)^n}^2, \quad c_0, c_1 > 0
\]

Proof. In fact

\[
\pi(t; y, y) \\
- \sum_{i,j=1}^{n} a_{ij}(x,t) y_i y_j dx
\]

\[
\geq \sum_{i,j=1}^{n} \left[ \lambda_i(a_i) - n \right] \int_{\Omega} |\nabla y_i|^2 \ dx
\]

From (10), we have

\[
\alpha \left[ \sum_{i=1}^{n} \int_{\Omega} |\nabla y_i|^2 \ dx \right] \geq c_1 \left\| y \right\|_{H_0^1(\Omega)^n}^2
\]

Adding \( \left\| y \right\|_{L^2(\Omega)^n}^2 \) to two sides, then we get the desired result.
We can now apply the results of [10, p.33] to obtain the following theorem:

**Theorem 1:**
If (8) and (10) hold, then there exist a unique solution \(y \in H^{2,1}(\Omega) \cap H^1(0,T; L^2(\Omega))\) satisfying the following system:

\[
\frac{\partial y}{\partial t} = (A(t)y) + f_i, \quad f_i \in L^2(Q) \quad \text{in } Q, \\
y_i(x,0) = u_i(x), \quad u_i(x) \in H^0(\Omega) \quad \text{in } \Omega, \\
y_i(x,t) = 0 \quad \text{on } \Sigma.
\]

Moreover, \(y\) is continuous from \([0,T]\) to \((H^1_0(\Omega))^n\).

**3. MINIMUM TIME AND CONTROLLABILITY IN SOBOLEV SPACE**

We denote the unique solution of (11), at time \(t\) for each control \(u = (u_1, u_2, \ldots, u_n)\) by \(y(t; u)\) and we write \(y(x,t; u)\) when the explicit dependence on \(x\) if necessary. Now we define the time-optimal control problem corresponding to the co-operative parabolic system (11):

\[
\min \{t : y(t; u) \in K^a, u \in U^a\},
\]

with constraints

\[
\begin{align*}
y(t; u) & \text{ is the solution of (11),} \\
U^a & = \{u = (u_1, u_2, \ldots, u_n) \in (H^1_0(\Omega))^n : \\
& \|u_i\|_{H^1_0(\Omega)} \leq \varepsilon\}, \\
K^a & = \{z = (z_1, z_2, \ldots, z_n) \in (L^2(\Omega))^n : \\
& \|z - z_i\|_{L^2(\Omega)} + \sum_{j=1}^n \left\| \frac{\partial z_i}{\partial x_j} - z_{i,j} \right\|_{L^2(\Omega)} \leq \varepsilon\},
\end{align*}
\]

and \(\varepsilon, \varepsilon > 0\) and \(z_{i,j} \in L^2(\Omega)\) are given.

**Theorem 2:**
If (8) and (10) are hold then the system (11) is controllable.

i.e., there exists \(\tau \in [0,T]\) and \(u \in U^a\) with \(y(\tau; u) \in K^a\).

**Proof:**
We can reduce the problem of controllability to the case of the system (11) with \(f_i = 0\). Here \(y(\tau; u) \in (H^1_0(\Omega))^n\). To show the system is controllable let \(\psi(x) \in H^{-1}(\Omega)\) such that

\[
< \psi(x), y(x, \tau; u) > = 0 \quad \forall u \in (H^1_0(\Omega))^n,
\]

where \(<, >\) denotes the duality paring between \(H^{-1}(\Omega)\) and \(H^1_0(\Omega)\).

Let us introduce the adjoint state \(p(t; u)\) by the solution of the following system

\[
-\frac{\partial p}{\partial t} - (A'(t)p(t; u)) = 0 \quad \text{in } \Omega \times [0,\tau], \\
p_i(x,0) = \bar{z}_i(x) \in H^{-1}(\Omega) \quad \text{in } \Omega, \\
p_i(x,t) = 0 \quad \text{in } \Gamma \times [0,\tau],
\]

where \(A'(t)\) is the adjoint of \(A(t)\) which is defined by

\[
< A'(t)\phi, \psi > = < \phi, A(t)\psi >, \quad \phi, \psi \in (H^1_0(\Omega))^n.
\]

Since \(\bar{z}_i \in H^{-1}(\Omega)\), the existence of a unique weak solution \(p\) for (15) can be proved by the transposition (see Chapter 3 in [9]). Multiplying the first equation in (15) by \(y_i(t; u)\) and integrate by parts from 0 to \(\tau\), we obtain the following identity;

\[
\int_0^\tau \int p_i(x,t; u)u_i(x,t; u)dxdt \leq < \bar{z}_i(x), y_i(x, \tau; u) > = 0.
\]

But from the continuity property, \(p_i(\tau; u) \equiv 0\) and hence \(\bar{z}_i = 0\).

Now set

\[
\tau^0 = \inf \{\tau : y(\tau; u) \in K^a \text{ for some } u \in U^a\}.
\]

Then, the following result holds.

**Theorem 3:**
If (8) and (10) are hold, then there exists an admissible control \(u^0\) to the problem (12)-(16), which
steering \( y(t;u^0) \) to hit a target set \( K^n_e \) in minimum time \( \tau^0 \) (defined by (16)). Moreover

\[
\sum_{i=1}^n (I_{n} - \Delta)(y_i(\tau^0;u^0) - z_{id})
\]

(17)

Proof:

Fixe \( x \), we can choose \( \tau^m \to \tau^0 \) and admissible controls \( \{u^m\} \) such that

\[
y(\tau^m;u^m) \in K^n_e, \quad m = 1,2,\ldots
\]

Set \( y^m = y(u^m) \). Since \( U^n_e \) is bounded, we may verify that \( y^m \) ranges in a bounded set in

\[
(L^2(0,T;L^2(\Omega)^n)) = (L^2(Q)^n).
\]

Again We may extract a subsequence denoted by \( \{u^m, y^m\} \) such that

\[
\begin{align*}
&u^m \to u^0 \quad \text{weakly in} \quad (H^1(\Omega)^n) \quad (u^0 \in U^n_e), \\
y^m \to y \quad \text{weakly in} \quad L^2(0,T;H^1(\Omega)^n).
\end{align*}
\]

(18)

We deduce from the equality

\[
\frac{dy^m}{dt} = f - A(t)y^m
\]

that is

\[
\frac{dy^m}{dt} \to \frac{dy}{dt} = f - A(t)y \quad \text{in} \quad L^2(0,T;H^{-1}(\Omega)^n).
\]

And

\[
y^m(0) \to y(0) = u^0 \quad \text{in} \quad U^n_e.
\]

But

\[
y(\tau^m;u^m) - y(\tau^0;u^0) = y(\tau^m;u^m) - y(\tau^0;u^m) + y(\tau^0;u^m) - y(\tau^0;u^0)
\]

and

\[
\begin{align*}
\|y(\tau^m;u^m) - y(\tau^0;u^0)\|_{H^{-1}(\Omega)^n} &= \left\| \int_0^{\tau^m} \frac{d}{dt} y(t;u^m) \, dt \right\|_{H^{-1}(\Omega)^n} \\
&\leq \sqrt{\tau^m - \tau^0} \left( \int_0^{\tau^m} \left\| \frac{d}{dt} y(t;u^m) \right\|_{H^{-1}(\Omega)^n} \, dt \right)^{1/2} \\
&\leq c \sqrt{\tau^m - \tau^0}
\end{align*}
\]

(20)

Combining (19) and (20), we get

\[
y(\tau^m;u^m) - y(\tau^0;u^0) \to 0 \quad \text{weakly in} \quad (H^{-1}(\Omega))^n.
\]

(21)

Since \( y(\tau^0;u^0) \in K^n_e \) as \( K^n_e \) is closed and convex, hence weakly closed. This shows that \( K^n_e \) is reached in time \( \tau^0 \) by the admissible control \( u^0 \).

For the second part of the theorem, really, from Theorem 1, the mapping \( t \to y(t;u) \) from \( [0,T] \to (H^1(\Omega)^n) \) is continuous for each fixed \( u \) and so \( y(\tau^0;u) \in \text{int} K^n_e \), for any \( u \in U^n_e \), by the minimality of \( \tau^0 \).

Using Theorem 1 it is easy to verify that the mapping \( u \to y(\tau^0;u) \), defined on \( (H^1(\Omega)^n) \), is continuous and linear. then, the set

\[
A(\tau^0) = \{y(\tau^0;u) | u \in U^n_e\}
\]

is the image under a linear mapping of a convex set, hence

\[
A(\tau^0) \quad \text{is convex. Thus we have}
\]

\[
A(\tau^0) \cap \text{int} K^n_e = \emptyset
\]

and \( y(\tau^0;u^0) \in \partial K^n_e \) (the boundary of \( K^n_e \)). Since \( \text{int} K^n_e \neq \emptyset \) from (14) there exists a closed hyperplane separating \( A(\tau^0) \) and \( K^n_e \) containing \( y(\tau^0;u^0) \), i.e., there is a nonzero \( g \in (L^2(\Omega)^n)^* \) such as

\[
< g, y(\tau^0;u) > \leq < g, y(\tau^0;u^0) > \leq \inf_{y \in K^n_e} < g, y(\tau^0;u) >
\]

(22)
From the second inequality in (22), \( g \) must support the set \( K_\varepsilon^n \) at \( y(x_0^0; u) \) i.e.,
\[
< g, (y(x_0^0; u) - y(x_0^0; y^0)) > \geq 0 \quad \forall u \in U^n_\varepsilon
\]
and since \((H_0^1(\Omega))^n\) is a Hilbert space, \( g \) must be of the form
\[
g = \lambda (y(x_0^0; u^0) - z_{id}) \quad \text{forsome } \lambda > 0.
\]
Dividing the inequality (22) by \( \lambda \) gives the desired result.

Now the inequality (17) can be interpreted as follows: let us introduce the adjoint state \( y(t; u) \) by the solution of the following system
\[
\begin{align*}
\frac{\partial y_1}{\partial t} - \Delta y_1 &= a_{11}(x,t)y_1 + a_{12}(x,t)y_2 + f_1, \\
x \in \Omega, \quad t \in [0, \tau^0], \\
\frac{\partial y_2}{\partial t} - \Delta y_2 &= a_{21}(x,t)y_1 + a_{22}(x,t)y_2 + f_2, \\
x \in \Omega, \quad t \in [0, \tau^0], \\
y_1(x,0) &= u_1^0(x), \quad y_2(x,0) = u_2^0(x), \\
x \in \Omega, \quad t \in [0, \tau^0],
\end{align*}
\]
with \( a_1(x,t), i, j = 1,2 \) are positive functions in \( L^n(\Omega) \),
\[
\lambda_1(a_{11}) \geq 2, \quad \lambda_1(a_{22}) \geq 2.
\]
The state \( y = (y_1, y_2) \) is solution of the following equations
\[
\begin{align*}
\int_\Omega (I - \Delta)(y_1(x, \tau^0; u^0) - z_{id}) \\
(y_1(x, \tau^0; u) - y_1(x, \tau^0; u^0))dx \\
= \int_\Omega p_1(0; u^0)(u - u^0)dx.
\end{align*}
\]
Hence the inequality (17) becomes
\[
\sum_{j=1}^n p_j(x,0; u^0)(u - u^0)dx \geq 0 \quad \forall u \in U^n_\varepsilon.
\]

Using the controllability condition (14), we have that the backward uniqueness property implies \( p_j(x,0; u^0) = 0 \). Hence the optimal control is bang-bang, i.e., \( \| u^0_i \|_{L^2(\Omega)}^2 = \varepsilon \) and since \( U^n_\varepsilon \) is strictly convex, the optimal control is unique. We have thus proved:

**Theorem 4:**

If (8) and (10) are hold, then there exists the adjoint state \( p \in L^2(0, \tau^0; (L^2(\Omega))^n) \) such that the optimal control \( u^0 \) of problem (12)-(16) is bang-bang and unique, which is determined by (23) and (24) together with (11) (with \( u_i = u_i^0, i = 1,2,\ldots,n \)).

**4. SCALAR CASE**

Here, we take the case where \( n = 2 \), the time-optimal problem is
\[
\min \{ t : y(x,t; u) \in K^2_\varepsilon, \ u = (u_1, u_2) \in U^n_\varepsilon \}.
\]
The state \( y = (y_1, y_2) \) is solution of the following equations
\[
\begin{align*}
\frac{\partial y_1}{\partial t} - \Delta y_1 &= a_{11}(x,t)y_1 + a_{12}(x,t)y_2 + f_1, \\
x \in \Omega, \quad t \in [0, \tau^0], \\
\frac{\partial y_2}{\partial t} - \Delta y_2 &= a_{21}(x,t)y_1 + a_{22}(x,t)y_2 + f_2, \\
x \in \Omega, \quad t \in [0, \tau^0], \\
y_1(x,0) &= u_1^0(x), \quad y_2(x,0) = u_2^0(x), \\
x \in \Omega, \quad t \in [0, \tau^0],
\end{align*}
\]
with
\[
\begin{align*}
a_{11}(x,t), i, j = 1,2 \text{ are positive functions in } L^n(\Omega), \\
\lambda_1(a_{11}) \geq 2, \quad \lambda_1(a_{22}) \geq 2.
\end{align*}
\]
The adjoint is a solution of the following equations
\[
\begin{align*}
\int_\Omega (I - \Delta)(y_1(x, \tau^0; u^0) - z_{id}) \\
(y_1(x, \tau^0; u) - y_1(x, \tau^0; u^0))dx \\
= \int_\Omega p_1(0; u^0)(u - u^0)dx.
\end{align*}
\]
Hence the inequality (17) becomes
\[
\sum_{j=1}^n p_j(x,0; u^0)(u - u^0)dx \geq 0 \quad \forall u \in U^n_\varepsilon.
\]
\[
\int_0^1 \left[ p_1(x,0; u^0_i)(u_i - u^0_i) + p_i(x,0; u^0_i)(u_i - u^0_i) \right] \, dxdt \geq 0 \quad \forall u \in U_c.
\]

5. COMMENTS

- In this communication we remark that if we have chosen to treat a special systems involving Laplace operator, most of the results we described without any change on the results, to more general parabolic systems involving the following second order operators:

\[ L(x, \cdot) = \sum_{i,j=1}^n b_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{j=1}^n b_j(x) \frac{\partial}{\partial x_j} + b_0(x, \cdot) \]

with sufficiently smooth coefficients (in particular, \( b_{ij}, b_j, b_0 \in L^\infty(Q), b_j, b_0 > 0 \)) and under the Legendre-Hadamard ellipticity condition

\[ \sum_{i,j=1}^n \eta_i \eta_j \geq \sigma \sum_{i=1}^n \eta_i \quad \forall (x,t) \in Q, \]

for all \( \eta_i \in \mathbb{R} \) and some constant \( \sigma > 0 \).

In this case we replace the first eigenvalue of the Laplace operator by the first eigenvalue of the operator \( L \) (see [5]).

- If we have chosen to treat a co-operative parabolic systems with Dirichlet boundary conditions. The results can be extended to the case of \( n \times n \) co-operative parabolic system with Neumann boundary conditions: If we take \( H^1(\Omega) \) instead of \( H^1_0(\Omega) \), we have to replace the Dirichlet boundary conditions \( y_j = 0, p_i = 0 \) on the boundary by Neumann boundary conditions \( \frac{\partial y_j}{\partial V} = 0, \frac{\partial p_i}{\partial V} = 0 \) where \( V \) is the outward normal.

- The results in this paper, carry over to the fixed-time problem (Chapter 3 in [9])

\[
\text{minimize} \sum_{i=1}^n \int_{\Omega} \left| y_i(x, T; u) - z_{id}(x) \right|^2 \, dx, \quad T \text{ fixed},
\]

subject to (11) (except the trivial case, where \( z_{id}(x) = y_i(x, T; u) \forall i = 1, 2, \ldots, n \) for some admissible control \( u \)). This can be proven in an analogous manner, as the necessary and sufficient conditions for the optimality of this problem coincide with (11), (15) and (24) (with \( u_i = u^0_i, i = 1, 2, \ldots, n \)).

REFERENCES


