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Controllability of Second-Order Impulsive Stochastic Integrodifferential Evolution System

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***Abstract*—In this paper, we derive a set of sufficient conditions for the controllability of second-order impulsive stochastic integrodifferential evolution system with nonlocal conditions in Hilbert spaces. The results are established by using fixed point technique.**

***Index Terms*—blackControllability, Second-order stochastic evolution system, Fixed point.**
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1 Introduction

Second order differential equations arise in many areas of science and technology. In particular, second order differential and integrodifferential equations serve as an abstract formulation of many partial integrodifferential equations which arise in problems connected with the transverse motion of an extensible beam, the vibration of hinged bars and many other physical phenomena. For instance 1001[6], a mathematical model for the deflections of an extensible beam with hinged ends, kept at a constant distance was introduced by 1001[22] and given by

$$\frac{\partial^2 u}{\partial t^2}+q\frac{\partial^4 u}{\partial x^4}-\Big[m_0+m_1\int_0^L\Big|\frac{\partial u}{\partial x}\Big|^2dx\Big]\frac{\partial^2 u}{\partial x^2}=0,$$
$$u(0,t)=u(L,t)=u_{xx}(0,t)=u_{xx}(L,t)=0, \quad (1.1)$$

where m_0, m_1, L and q are positive real numbers, L is the beam's length, the nonlinear term represents the beam's tension due to its extensibility and $u(x,t)$ stands for the beam's deflection at point x and instant t. Also, the mathematical model below describes the small transverse deflections of an extensible beam with moving hinged ends and variable tension:

$$\frac{\partial^2 u}{\partial t^2}+q\frac{\partial^4 u}{\partial x^4}-\Big[\hat{a}(t)+\hat{b}(t)\int_{\alpha(t)}^{\beta(t)}\Big|\frac{\partial u}{\partial x}\Big|^2dx\Big]\frac{\partial^2 u}{\partial x^2}=0,$$
$$u(\alpha(t),t)=u(\beta(t),t)=u_{xx}(\alpha(t),t)=u_{xx}(\beta(t),t)=0, \quad (1.2)$$

where $\hat{a}(t)$ and $\hat{b}(t)$ are positive real functions, $[\alpha(t),\beta(t)]$ is the deformation of $[\alpha_0,\beta_0]$ after time $t>0$ with $\alpha_0=\alpha(0)$, $\beta_0=\beta(0)$, $0<\alpha(t)<\alpha_0<\beta_0<\beta(t)$ for all $t>0$. We can propose a natural mixed problem motivated by above models as in 1001[15] and the above equations can be written in the abstract form with general nonlinear term f as

$$u''(t)+A^2u(t)-M(\|A^{\frac{1}{2}}u(t)\|_H^2)Au(t)=f(t,u,u'), \quad (1.3)$$

where A is a linear operator in a Hilbert space H, M and f are real functions. If we take $M(\|A^{\frac{1}{2}}u\|_H^2)A-A^2=A$ for appropriate assumptions on M then (1.3) becomes

$$u''(t) = Au(t)+f(t,u(t),u'(t)), \quad (1.4)$$

Several papers 1001[16], [21] have been devoted to the study of existence of abstract second order differential equation (1.4). As pointed out in 1001[10], if experimentally there is variance in measurements, then it is advantageous to study a stochastic version of the model for understanding the effects of so-called noise on the behavior of the phenomenon. Hence the stochastic generalization of (1.4) is the following equation

$$\begin{aligned} du'(t) &= [Au(t)+f(t,u(t),u'(t))]dt+ \\ &\quad g(t,u(t),u'(t))dw(t), \\ u(0) &= u_0, \quad u'(0)=u_1. \end{aligned} \quad (1.5)$$

Many authors have been extensively discussed the controllability of second order deterministic and stochastic differential systems in 1001[2], [13], [14], [17]. Now, we extend our study to the second order stochastic evolution equation by using the fundamental solution of a second order evolution equations by Kozak 1001[11]. However, the study on existence and controllability of second order integrodifferential evolution systems has gained renewed interest and only few papers have appeared (see 1001[1], [3], [4]). Nowadays, there has been increasing interest in the analysis and synthesis of impulsive systems due to their significance both in theory and applications 1001[12], [20]. Further Sakthivel et al. 1001[18], [19] explored the controllability of second order deterministic and stochastic impulsive differential systems. Problems associated with non local conditions have many applications and the current analysis of such problems has drawn great attention (see 1001[8], [9]). Upto now, there is no work reported on controllability of second order impulsive stochastic integrodifferential evolution system which motivates our present work. Consider the following class of second order impulsive stochastic integrodifferential evolution system with nonlocal conditions

$$\begin{aligned} dx'(t) &= \Big[A(t)x(t)+Bu(t)+\int_0^t f\Big(t,s,x(s),x'(s), \\ &\quad \int_0^s g\big(s,\tau,x(\tau),x'(\tau)\big)d\tau\Big)ds\Big]dt+ \\ &\quad \sigma(t,x(t),x'(t))dw(t),\ t\in J:=[0,a], t\neq t_k, \\ x(0) &= x_0+p(x,x'), \qquad x'(0)=y_0+q(x,x'), \\ \triangle x(t_k) &= I_k(x(t_k),x'(t_k^-)), \\ \triangle x'(t_k) &= J_k(x(t_k),x'(t_k^-)),\ k=1,2,\cdots,m, \end{aligned} \quad (1.6)$$

where the state variable $x(\cdot)$ takes values in a real separable Hilbert space H with innerproduct $(\cdot,\cdot)$ and norm $\|\cdot\|$ and $A(t): H \to H$ is a closed densely defined operator on H. The control function $u(\cdot)$ takes values in $L^2(J,U)$ of admissible control functions for a separable Hilbert space U and B is a bounded linear operator from U into H. Let K be another separable Hilbert space with innerproduct $(\cdot,\cdot)_K$ and the norm $\|\cdot\|_K$. Suppose $\{w(t) : t \geq 0\}$ is a given K-valued Wiener process with a finite trace nuclear covariance operator $Q \geq 0$. We employ the same notation $\|\cdot\|$ for the norm $\mathcal{L}(K,H)$, where $\mathcal{L}(K,H)$ denotes the space of all bounded linear operators from K into H. Further, $f : \Lambda \times H \times H \times H \to H$, $g : \Lambda \times H \times H \to H$ and $\sigma : J \times H \times H \to \mathcal{L}_Q(K,H)$ are measurable mappings in H-norm and $\mathcal{L}_Q(K,H)$-norm respectively. Here $\mathcal{L}_Q(K,H)$ denotes the space of all Q-Hilbert-Schmidt operators from K into H which will be defined in Section 2 and $\Lambda = \{(t,s) \in J \times J : s \leq t\}$. The functions $p, q : \mathcal{PC}(J,H) \times \mathcal{PC}(J,H) \to H$ and $I_k, J_k \in C(H \times H, H)$ $(k = 1,2,\cdots,m)$ are bounded functions. Furthermore, the fixed times t_k satisfies $0 = t_0 < t_1 < t_2 < \cdots < t_m < t_{m+1} = a$ and $\triangle\xi(t_k) = \xi(t_k^+) - \xi(t_k^-)$ represents the jump of a function ξ at t_k where $\xi(t_k^+)$ and $\xi(t_k^-)$ denote the right and left limits of ξ at t_k and I_k, J_k determines the size of the jump. The initial values x_0, y_0 are $\mathcal{F}_0$-adapted, H-valued random variables independent of $\{w(t) : t \geq 0\}$ with finite second moment.

2. Preliminaries

Let $(\Omega, \mathcal{F}, P; \mathbf{F})$ $\{\mathbf{F} = \{\mathcal{F}_t\}_{t\geq 0}\}$ be a complete filtered probability space satisfying that $\mathcal{F}_0$ contains all P-null sets of $\mathcal{F}$. An H-valued random variable is an $\mathcal{F}$-measurable function $x(t)$: $\Omega \to H$ and the collection of random variables $S = \{x(t,\omega) : \Omega \to H \backslash t \in J\}$ is called a stochastic process. Generally, we just write $x(t)$ instead of $x(t,\omega)$ and $x(t) : J \to H$ in the space of S.

Let $\{e_i\}_{i=1}^\infty$ be a complete orthonormal basis of K. Suppose that $\{w(t) : t \geq 0\}$ is a cylindrical K-valued wiener process with a finite trace nuclear covariance operator $Q \geq 0$, denote $Tr(Q) = \sum_{i=1}^\infty \lambda_i = \lambda < \infty$, which satisfies that $Qe_i = \lambda_i e_i$. So, actually, $w(t) = \sum_{i=1}^\infty \sqrt{\lambda_i}\omega_i(t)e_i$, where $\{\omega_i(t)\}_{i=1}^\infty$ are mutually independent one-dimensional standard Wiener processes. We assume that $\mathcal{F}_t = \sigma\{w(s) : 0 \leq s \leq t\}$ is the σ-algebra generated by w and $\mathcal{F}_a = \mathcal{F}$. Let $\Psi \in \mathcal{L}(K,H)$ and define $\|\Psi\|_Q^2 = Tr(\Psi Q \Psi^*) = \sum_{i=1}^\infty \|\sqrt{\lambda_i}\Psi e_i\|^2$. If $\|\Psi\|_Q < \infty$, then Ψ is called a Q-Hilbert-Schmidt operator. Let $\mathcal{L}_Q(K,H)$ denote the space of all Q-Hilbert-Schmidt operators $\Psi : K \to H$. The completion $\mathcal{L}_Q(K,H)$ of $\mathcal{L}(K,H)$ with respect to the topology induced by the norm $\|\cdot\|_Q$ where $\|\Psi\|_Q^2 = \langle\Psi,\Psi\rangle$ is a Hilbert space with the above norm topology. For more details refer to Da Prato 1001[5].

The collection of all strongly measurable, square integrable H-valued random variables denoted by $\mathcal{L}_2(\Omega,\mathcal{F},P;H) \equiv \mathcal{L}_2(\Omega,H)$, is a Banach space equipped with norm $\|x(\cdot)\|_{\mathcal{L}_2} = (E\|x(\cdot;\omega)\|_H^2)^{\frac{1}{2}}$, where the expectation E is defined by $E(h) = \int_\Omega h(\omega)dP$. Similarly, $\mathcal{L}_2^{\mathcal{F}}(\Omega,H)$ denotes the Banach space of all $\mathcal{F}_t$-measurable, square integrable random variables, such that $\int_\Omega \|x(t,\cdot)\|_{\mathcal{L}_2}^2 dt < \infty$. Denote $J_0 = [0,t_1], J_k = (t_k,t_{k+1}], k = 1,2,\cdots,m$, and define the following class of functions:

$\mathcal{H}_2 = \mathcal{PC}(J,\mathcal{L}_2(\Omega,H)) = \{x : J \to \mathcal{L}_2 : x|_{(t_k,t_{k+1}]} \in \mathcal{C}(J_k,\mathcal{L}_2), k = 0,1,2,\cdots,m$ and there exist $x(t_k^-)$ and $x(t_k^+)$ with $x(t_k^-) = x(t_k)$, $k = 1,2,3,\cdots,m\}$ is the Banach space of all piecewise continuous maps from J into $\mathcal{L}_2(\Omega,H)$ provided with the norm

$$\|x\|_{\mathcal{H}_2}^2 = \sup_{t\in J} E\|x(t)\|^2.$$

$\mathcal{PC}^1(J,\mathcal{L}_2(\Omega,H)) = \{x \in \mathcal{H}_2 : x|_{(t_k,t_{k+1}]} \in \mathcal{C}^1(J_k,\mathcal{L}_2), k = 0,1,2,\cdots,m$ and there exist $x'(t_k^-)$ and $x'(t_k^+)$ with $x'(t_k^-) = x'(t_k)$, $k = 1,2,3,\cdots,m\}$ is the Banach space of all piecewise continuously differentiable maps from J into $\mathcal{L}_2(\Omega,H)$ satisfying $\sup_{t\in J} E\|x(t)\|^2 < \infty$, $\sup_{t\in J} E\|x'(t)\|^2 < \infty$. Let $\mathcal{H}_2' \equiv \mathcal{PC}^1(J,\mathcal{L}_2)$ be the closed subspace of $\mathcal{PC}^1(J,\mathcal{L}_2^{\mathcal{F}}(\Omega,H))$ consisting of measurable, $\mathcal{F}_t$-adapted and H-valued processes $x(t)$. Then H_2' is a Banach space endowed with the norm

$$\|x\|_{\mathcal{H}_2'}^2 = \|x\|_{\mathcal{H}_2}^2 + \|x'\|_{\mathcal{H}_2}^2$$

where $\|x\|_{\mathcal{H}_2}^2 = \sup_{t\in J} E\|x(t)\|^2$ and $\|x'\|_{\mathcal{H}_2}^2 = \sup_{t\in J} E\|x'(t)\|^2$. Let H denote a real separable Hilbert space and for each $t \in J$, let $A(t) : H \to H$ be a closed densely defined operator. The fundamental solution for second order evolution equation

$$x''(t) = A(t)x(t) \tag{2.1}$$

developed by Kozak 1001[11] is as follows. Let us assume that the domain of $A(t)$ does not depend on $t \in J$ and denote it by $D(A)$ (for each $t \in J$, $D(A(t)) = D(A)$).

Definition: 2.1. 1001[11] *A family $\mathcal{S}$ of bounded linear operators $S(t,s) : H \to H$, $t,s \in J$, is called fundamental solution of a second order equation* (2.1) *if:*

(Z1) For each $x \in H$ the mapping $J \times J \ni (t,s) \to S(t,s)x \in H$ is of class C^1
- (i) for each $t \in J$, $S(t,t) = 0$,
- (ii) for all $t,s \in J$ and for each $x \in H$, $\frac{\partial}{\partial t}S(t,s)\Big|_{t=s} x = x$, $\frac{\partial}{\partial s}S(t,s)\Big|_{t=s} x = -x$.

(Z2) For all $t,s \in J$, if $x \in D(A)$, then $S(t,s)x \in D(A)$, the mapping $J \times J \ni (t,s) \to S(t,s)x \in H$ is of class C^2 and
- (i) $\frac{\partial^2}{\partial t^2}S(t,s)x = A(t)S(t,s)x$,
- (ii) $\frac{\partial^2}{\partial s^2}S(t,s)x = S(t,s)A(s)x$,
- (iii) $\frac{\partial}{\partial s}\frac{\partial}{\partial t}S(t,s)\Big|_{t=s} x = 0$.

(Z3) For all $t,s \in J$, if $x \in D(A)$, then $\frac{\partial}{\partial s}S(t,s)x \in D(A)$, there exist $\frac{\partial^2}{\partial t^2}\frac{\partial}{\partial s}S(t,s)x$, $\frac{\partial^2}{\partial s^2}\frac{\partial}{\partial t}S(t,s)x$, and
- (i) $\frac{\partial^2}{\partial t^2}\frac{\partial}{\partial s}S(t,s)x = A(t)\frac{\partial}{\partial s}S(t,s)x$,
- (ii) $\frac{\partial^2}{\partial s^2}\frac{\partial}{\partial t}S(t,s)x = \frac{\partial}{\partial t}S(t,s)A(s)x$ and the mapping $J \times J \ni (t,s) \to A(t)\frac{\partial}{\partial s}S(t,s)x$ is continuous.

Definition: 2.2. 1001[7] *A stochastic process x is said to be a mild solution of* (1.6) *if the following conditions are satisfied:*

(a) $x(t,\omega)$ *is a measurable function from* $J \times \Omega$ *to* H *and* $x(t)$ *is* $\mathcal{F}_t$*-adapted for all* $t \in J$,
(b) $E\|x(t)\|^2 < \infty$, $E\|x'(t)\|^2 < \infty$, *for each* $t \in J$,
(c) $\triangle x(t_k) = I_k(x(t_k), x'(t_k^-))$, $\triangle x'(t_k) = J_k(x(t_k), x'(t_k^-))$, $k = 1, 2, \cdots, m$,
(d) *For each* $u \in L_2^{\mathcal{F}}(J, U)$ *the process x satisfies the following integral equation:*

$$\begin{aligned} x(t) &= -\frac{\partial}{\partial s}S(t,s)\Big|_{s=0}[x_0 + p(x,x')] + S(t,0)[y_0 + q(x,x')] + \int_0^t S(t,s)Bu(s)ds \\ &\quad + \int_0^t S(t,s)\Big[\int_0^s f\Big(s,\tau,x(\tau),x'(\tau),\int_0^\tau g\big(\tau,\eta,x(\eta),x'(\eta)\big)d\eta\Big)d\tau\Big]ds \\ &\quad + \int_0^t S(t,s)\sigma(s,x(s),x'(s))dw(s) - \sum_{0<t_k<t}\frac{\partial}{\partial s}S(t,s)\Big|_{s=t_k} I_k(x(t_k), x'(t_k^-)) \\ &\quad + \sum_{0<t_k<t} S(t,t_k)J_k(x(t_k), x'(t_k^-)), \quad \textit{for a.e. } t \in J, \\ x(0) &= x_0 + p(x,x'),\ x'(0) = y_0 + q(x,x'). \end{aligned} \tag{2.2}$$

Definition: 2.3. *The system* (1.6) *is said to be controllable on the interval J, if for every $x_0, y_0 \in D(A)$ and $x_1 \in H$, there exists a control $u \in L^2(J,U)$ such that the solution $x(\cdot)$ of* (1.6) *satisfies $x(a) = x_1$.*

Before proceeding to the main result we shall set forth a list of hypotheses:

(H1) $x(t) \in D(A(t))$, for each $t \in J$.
(H2) There exists a fundamental solution $S(t,s)$ of (2.1).
(H3) For each $t, s \in J$ there exist positive constants M, M^* and N, N^* such that

$$M = \sup\{\|S(t,s)\|^2 : t,s \in J\},\ M^* = \sup\{\Big\|\frac{\partial}{\partial s}S(t,s)\Big\|^2 : t,s \in J\} \text{ and}$$

$$N = \sup\{\Big\|\frac{\partial}{\partial t}S(t,s)\Big\|^2 : t,s \in J\},\ N^* = \sup\{\Big\|\frac{\partial}{\partial t}\frac{\partial}{\partial s}S(t,s)\Big\|^2 : t,s \in J\} \text{ respectively.}$$

(H4) The linear operator $W : L^2(J,U) \to H$ defined by

$$Wu = \int_0^a S(a,s)Bu(s)ds$$

is invertible with inverse operator W^{-1} taking values in $L^2(J,U) \setminus kerW$ and there exist positive constants M_B, M_W such that $\|B\|^2 \le M_B,\ \|W^{-1}\|^2 \le M_W$.

(H5) The nonlinear function $f : \Lambda \times H \times H \times H \to H$ is continuous and there exist constants $M_f > 0$, $\tilde{M}_f > 0$ for $(t,s) \in \Lambda$ and $x_1, x_2, x_3, y_1, y_2, y_3 \in H$ such that

$$E\Big\|\int_0^t [f(t,s,x_1,x_2,x_3) - f(t,s,y_1,y_2,y_3)]ds\Big\|^2 \le M_f[\|x_1 - y_1\|^2 + \|x_2 - y_2\|^2 + \|x_3 - y_3\|^2]$$

and $\tilde{M}_f = \sup_{(t,s)\in\Lambda} \| \int_0^t f(t,s,0,0,0)ds\|^2$.

(H6) The nonlinear function $g : \Lambda \times H \times H \to H$ is continuous and there exist positive constants M_g, $\tilde{M}_g$, for $x_1, y_1, x_2, y_2 \in H$ and $(t,s) \in \Lambda$ such that

$$E\Big\|\int_0^t \big[g(t,s,x_1,x_2) - g(t,s,y_1,y_2)\big]ds\Big\|^2 \le M_g[\|x_1 - y_1\|^2 + \|x_2 - y_2\|^2]$$

and $\tilde{M}_g = \sup_{(t,s)\in\Lambda} \| \int_0^t g(t,s,0,0)ds\|^2$.

(H7) The function $\sigma : J \times H \times H \to \mathcal{L}_Q(K,H)$ is continuous and there exist constants $M_\sigma > 0$, $\tilde{M}_\sigma > 0$ for $t \in J$ and $x_1, y_1, x_2, y_2 \in H$ such that

$$E\|\sigma(t,x_1,x_2) - \sigma(t,y_1,y_2)\|_Q^2 \le M_\sigma[\|x_1 - y_1\|^2 + \|x_2 - y_2\|^2]$$

and $\tilde{M}_\sigma = \sup_{t\in J} \|\sigma(t,0,0)\|^2$.

(H8) The functions $p, q : \mathcal{PC}(J,H) \times \mathcal{PC}(J,H) \to H$ are continuous and there exist positive constants $M_p, \tilde{M}_p$, $M_q, \tilde{M}_q$ for $x_1, y_1, x_2, y_2 \in \mathcal{PC}(J,H)$ such that

$$E\|p(x_1,x_2) - p(y_1,y_2)\|^2 \le M_p[\|x_1 - y_1\|^2 + \|x_2 - y_2\|^2],\quad E\|p(x_1,x_2)\|^2 \le \tilde{M}_p,$$

$$E\|q(x_1,x_2) - q(y_1,y_2)\|^2 \le M_q[\|x_1 - y_1\|^2 + \|x_2 - y_2\|^2],\quad E\|q(x_1,x_2)\|^2 \le \tilde{M}_q.$$

(H9) $I_k, J_k : H\times H \to H$ are continuous and there exist constants $\beta_k,\ \tilde{\beta}_k,\ \alpha_k,\ \tilde{\alpha}_k > 0$ such that

$$E\|I_k(x_1,x_2)-I_k(y_1,y_2)\|^2 \le \beta_k[\|x_1-y_1\|^2+\|x_2-y_2\|^2],\quad k=1,2,\cdots,m,$$

$$E\|J_k(x_1,x_2)-J_k(y_1,y_2)\|^2 \le \alpha_k[\|x_1-y_1\|^2+\|x_2-y_2\|^2],\quad k=1,2,\cdots,m$$

and $\tilde{\beta_k}=\|I_k(0,0)\|^2, \tilde{\alpha_k}=\|J_k(0,0)\|^2,\ k=1,2,\cdots,m.$

(H10) There exists a constant $r>0$ such that
$9\Big[(M^*+N^*)\Big(\|x_0\|^2+\tilde{M}_p+2m\big[\sum_{k=1}^m\beta_k r+\sum_{k=1}^m\tilde{\beta}_k\big]\Big)+(M+N)\Big(\|y_0\|^2+\tilde{M}_q$
$+a^2M_B\mathcal{G}+2a^2[M_f((1+2M_g)r+2\tilde{M}_g)+\tilde{M}_f]+2aTr(Q)[M_\sigma r+\tilde{M}_\sigma]$
$+2m\big[\sum_{k=1}^m\alpha_k r+\sum_{k=1}^m\tilde{\alpha}_k\big]\Big)\Big]\le r$
and $\nu=7\Big\{(\gamma M^*+\delta N^*)\Big[M_p+m\sum_{k=1}^m\beta_k\Big]+(\gamma M+\delta N)\Big[M_q+a^2M_f(1+M_g)$
$+aTr(Q)M_\sigma\ +m\sum_{k=1}^m\alpha_k\Big]\Big\}$
where $\gamma=1+6a^2MM_BM_W,\quad \delta=1+6a^2NM_BM_W.$

3 Controllability Result

Theorem: 3.1. *If the conditions* $(H1)-(H10)$ *are satisfied and if* $0\le\nu<1$*, then the system* (1.6) *is controllable on* J.

Proof: For an arbitrary function $x(\cdot)$ we define the control using the hypothesis $(H2)$

$$\begin{aligned} u(t) &= W^{-1}\Big[x_1+\frac{\partial}{\partial s}S(a,s)\Big|_{s=0}[x_0+p(x,x')]-S(a,0)[y_0+q(x,x')]\\ &-\int_0^a S(a,s)\Big[\int_0^s f\Big(s,\tau,x(\tau),x'(\tau),\int_0^\tau g\big(\tau,\eta,x(\eta),x'(\eta)\big)d\eta\Big)d\tau\Big]ds\\ &-\int_0^a S(a,s)\sigma(s,x(s),x'(s))dw(s)+\sum_{0<t_k<a}\frac{\partial}{\partial s}S(a,s)\Big|_{s=t_k}I_k(x(t_k),x'(t_k^-))\\ &-\sum_{0<t_k<a}S(a,t_k)J_k(x(t_k),x'(t_k^-))\Big](t). \end{aligned} \tag{3.1}$$

Let $\mathcal{Y}_r$ be a nonempty closed subset of $\mathcal{H}_2'$ defined by

$$\mathcal{Y}_r=\{x: x\in\mathcal{H}_2' | E\|x(t)\|^2_{\mathcal{H}_2'}\le r\}.$$

Consider the mapping $\Phi:\mathcal{Y}_r\to\mathcal{Y}_r$ defined by

$$\begin{aligned} (\Phi x)(t) &= -\frac{\partial}{\partial s}S(t,s)\Big|_{s=0}[x_0+p(x,x')]+S(t,0)[y_0+q(x,x')]+\int_0^t S(t,s)BW^{-1}\Big[x_1\\ &+\frac{\partial}{\partial s}S(a,s)|_{s=0}[x_0+p(x,x')]-S(a,0)[y_0+q(x,x')]\\ &-\int_0^a S(a,s)\Big[\int_0^s f\Big(s,\tau,x(\tau),x'(\tau),\int_0^\tau g\big(\tau,\eta,x(\eta),x'(\eta)\big)d\eta\Big)d\tau\Big]ds\\ &-\int_0^a S(a,s)\sigma(s,x(s),x'(s))dw(s)+\sum_{0<t_k<a}\frac{\partial}{\partial s}S(a,s)\Big|_{s=t_k}I_k(x(t_k),x'(t_k^-))\\ &-\sum_{0<t_k<a}S(a,t_k)J_k(x(t_k),x'(t_k^-))\Big](s)ds+\int_0^t S(t,s)\sigma(s,x(s),x'(s))dw(s)\\ &+\int_0^t S(t,s)\Big[\int_0^s f\Big(s,\tau,x(\tau),x'(\tau),\int_0^\tau g\big(\tau,\eta,x(\eta),x'(\eta)\big)d\eta\Big)d\tau\Big]ds\\ &-\sum_{0<t_k<t}\frac{\partial}{\partial s}S(t,s)\Big|_{s=t_k}I_k(x(t_k),x'(t_k^-))+\sum_{0<t_k<t}S(t,t_k)J_k(x(t_k),x'(t_k^-)) \end{aligned}$$

for $t \in J$ and also we have

$$\begin{aligned}(\Phi x)'(t) &= -\frac{\partial}{\partial t}\frac{\partial}{\partial s}S(t,s)\Big|_{s=0}[x_0+p(x,x')]+\frac{\partial}{\partial t}S(t,0)[y_0+q(x,x')]+\int_0^t\frac{\partial}{\partial t}S(t,s)\times\\ &\times BW^{-1}\Big[x_1+\frac{\partial}{\partial s}S(a,s)\Big|_{s=0}[x_0+p(x,x')]-S(a,0)[y_0+q(x,x')]\\ &-\int_0^a S(a,s)\Big[\int_0^s f\Big(s,\tau,x(\tau),x'(\tau),\int_0^\tau g\big(\tau,\eta,x(\eta),x'(\eta)\big)d\eta\Big)d\tau\Big]ds\\ &-\int_0^a S(a,s)\sigma(s,x(s),x'(s))dw(s)+\sum_{0<t_k<a}\frac{\partial}{\partial s}S(a,s)\Big|_{s=t_k}I_k(x(t_k),x'(t_k^-))\\ &-\sum_{0<t_k<a}S(a,t_k)J_k(x(t_k),x'(t_k^-))\Big](s)ds+\int_0^t\frac{\partial}{\partial t}S(t,s)\sigma(s,x(s),x'(s))dw(s)\\ &+\int_0^t\frac{\partial}{\partial t}S(t,s)\Big[\int_0^s f\Big(s,\tau,x(\tau),x'(\tau),\int_0^\tau g\big(\tau,\eta,x(\eta),x'(\eta)\big)d\eta\Big)d\tau\Big]ds\\ &-\sum_{0<t_k<t}\frac{\partial}{\partial t}\frac{\partial}{\partial s}S(t,s)\Big|_{s=t_k}I_k(x(t_k),x'(t_k^-))+\sum_{0<t_k<t}\frac{\partial}{\partial t}S(t,t_k)J_k(x(t_k),x'(t_k^-)).\end{aligned}$$

It is easy to see that Φ is continuous in $\mathcal{H}_2'$ since all the functions involved in the operator Φ, Φ' are continuous. Using the above control, we have to show that the operator Φ has a fixed point. Clearly $(\Phi x)(a) = x_1$ which means that the control u steers the system from the initial state x_0 to x_1 in time a. From our assumptions we evaluate

$$\begin{aligned}E\|u_x(t)\|^2 &\leq 9M_W\Big[\|x_1\|^2+M^*\|x_0\|^2+M^*\tilde{M}_p+M\|y_0\|^2+M\tilde{M}_q+2a^2M\times\\ &\times[M_f((1+2M_g)r+2\tilde{M}_g)+\tilde{M}_f]+2aMTr(Q)[M_\sigma r+\tilde{M}_\sigma]\\ &+2mM^*\big[\sum_{k=1}^m\beta_k r+\sum_{k=1}^m\tilde{\beta}_k\big]+2mM\big[\sum_{k=1}^m\alpha_k r+\sum_{k=1}^m\tilde{\alpha}_k\big]\Big]:=\mathcal{G}\text{ and}\\ E\|u_x(t)-u_y(t)\|^2 &\leq 6M_W\Big[M\Big(M_q+a^2M_f(1+M_g)+aTr(Q)M_\sigma+m\sum_{k=1}^m\alpha_k\Big)\\ &+M^*\Big(M_p+m\sum_{k=1}^m\beta_k\Big)\Big][\|x-y\|^2+\|x'-y'\|^2].\end{aligned}$$

Step 1: The operator Φ maps $\mathcal{Y}_r$ into $\mathcal{Y}_r$.

$$\begin{aligned}E\|\Phi x(t)\|^2 &\leq 9\Big[M^*\Big(\|x_0\|^2+\tilde{M}_p+2m\big[\sum_{k=1}^m\beta_k r+\sum_{k=1}^m\tilde{\beta}_k\big]\Big)+M\Big(\|y_0\|^2+\tilde{M}_q+a^2M_B\mathcal{G}\\ &+2a^2[M_f((1+2M_g)r+2\tilde{M}_g)+\tilde{M}_f]+2aTr(Q)[M_\sigma r+\tilde{M}_\sigma]\\ &+2m\big[\sum_{k=1}^m\alpha_k r+\sum_{k=1}^m\tilde{\alpha}_k\big]\Big)\Big]\quad\text{and}\end{aligned}\tag{3.2}$$

$$\begin{aligned}E\|(\Phi x)'(t)\|^2 &\leq 9\Big[N^*\Big(\|x_0\|^2+\tilde{M}_p+2m\big[\sum_{k=1}^m\beta_k r+\sum_{k=1}^m\tilde{\beta}_k\big]\Big)+N\Big(\|y_0\|^2+\tilde{M}_q+a^2M_B\mathcal{G}\\ &+2a^2[M_f((1+2M_g)r+2\tilde{M}_g)+\tilde{M}_f]+2aTr(Q)[M_\sigma r+\tilde{M}_\sigma]\\ &+2m\big[\sum_{k=1}^m\alpha_k r+\sum_{k=1}^m\tilde{\alpha}_k\big]\Big)\Big].\end{aligned}\tag{3.3}$$

Hence adding (3.2) and (3.3) yields

$$\begin{aligned}E\|\Phi x(t)\|^2_{\mathcal{H}_2'} &\leq 9\Big[(M^*+N^*)\Big(\|x_0\|^2+\tilde{M}_p+2m\big[\sum_{k=1}^m\beta_k r+\sum_{k=1}^m\tilde{\beta}_k\big]\Big)+(M+N)\Big(\|y_0\|^2\\ &+\tilde{M}_q+a^2M_B\mathcal{G}+2a^2[M_f((1+2M_g)r+2\tilde{M}_g)+\tilde{M}_f]+2aTr(Q)[M_\sigma r\\ &+\tilde{M}_\sigma]+2m\big[\sum_{k=1}^m\alpha_k r+\sum_{k=1}^m\tilde{\alpha}_k\big]\Big)\Big].\end{aligned}$$

From $(H10)$ we obtain $E\|(\Phi x)(t)\|^2 \leq r$. Hence Φ maps $\mathcal{Y}_r$ into itself.
Step 2: $\Phi : \mathcal{Y}_r \to \mathcal{Y}_r$ is a contraction mapping.
Let $x, y \in \mathcal{Y}_r$ and we evaluate

$$\begin{aligned}
&E\|(\Phi x)(t) - (\Phi y)(t)\|^2 \\
&\leq E\Big\|\frac{\partial}{\partial s}S(t,s)\Big|_{s=0}[p(x,x') - p(y,y')]\Big\|^2 + E\|S(t,0)[q(x,x') - q(y,y')]\|^2 \\
&+E\Big\|\int_0^t S(t,s)\Big[\int_0^s f\Big(s,\tau,x(\tau),x'(\tau),\int_0^\tau g(\tau,\eta,x(\eta),x'(\eta))d\eta\Big)d\tau \\
&-f\Big(s,\tau,y(\tau),y'(\tau),\int_0^\tau g(\tau,\eta,y(\eta),y'(\eta))d\eta\Big)d\tau\Big]ds\Big\|^2 + E\Big\|\int_0^t S(t,s)B[u_x(s) \\
&-u_y(s)]ds\Big\|^2 + E\Big\|\int_0^t S(t,s)[\sigma(s,x(s),x'(s)) - \sigma(s,y(s),y'(s))]dw(s)\Big\|^2 \\
&+E\Big\|\sum_{0<t_k<t}\frac{\partial}{\partial s}S(t,s)\Big|_{s=t_k}[I_k(x(t_k),x'(t_k^-)) - I_k(y(t_k),y'(t_k^-))]\Big\|^2 \\
&+E\Big\|\sum_{0<t_k<t}S(t,t_k)[J_k(x(t_k),x'(t_k^-)) - J_k(y(t_k),y'(t_k^-))]\Big\|^2 \\
&\leq 7\Big\{(1+6a^2MM_BM_W)(M^*M_p + MM_q + a^2MM_f(1+M_g) + aMTr(Q)M_\sigma \\
&\quad +mM^*\sum_{k=1}^m\beta_k + mM\sum_{k=1}^m\alpha_k)\Big\}[\|x-y\|^2 + \|x'-y'\|^2] \quad \text{and} \qquad (3.4)
\end{aligned}$$

$$\begin{aligned}
E\|(\Phi x)'(t)-(\Phi y)'(t)\|^2 \leq 7\Big\{&(1+6a^2NM_BM_W)(N^*M_p+NM_q+a^2NM_f(1+M_g)+aN\times \\
&\times Tr(Q)M_\sigma+mN^*\sum_{k=1}^m\beta_k+mN\sum_{k=1}^m\alpha_k)\Big\}[\|x-y\|^2 + \|x'-y'\|^2]. \qquad (3.5)
\end{aligned}$$

Hence adding (3.4) and (3.5) gives

$$\begin{aligned}
E\|(\Phi x)(t)\|^2_{\mathcal{H}_2'} \quad \leq \quad & 7\Big\{(\gamma M^* + \delta N^*)\Big[M_p + m\sum_{k=1}^m\beta_k\Big] + (\gamma M + \delta N)\Big[M_q + a^2M_f(1+M_g) \\
&+aTr(Q)M_\sigma + m\sum_{k=1}^m\alpha_k\Big]\Big\}\|x-y\|^2_{\mathcal{H}_2'} \leq \nu\|x-y\|^2_{\mathcal{H}_2'}.
\end{aligned}$$

Since $\nu < 1$, the mapping Φ is a contraction and hence by Banach fixed point theorem there exists a unique fixed point $x \in \mathcal{Y}_r$ such that $(\Phi x)(t) = x(t)$. This fixed point is then the solution of the system (1.6) and clearly, $x(a) = (\Phi x)(a) = x_1$ which implies that the system (1.6) is controllable on J. □

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