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Numerical Verification of the Lagrange's Mean Value Theorem using MATLAB

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ABSTRACT

In this paper, we present numerical exploration of Lagrange's Mean Value Theorem. It considers a representative group of functions in order to determine in the first place, a straight line that averages the value of the integral and second for some of these same functions but within an interval, the tangent straight lines are generated. For both of these situations, Matlab software is used to obtain their derivative and roots in order to know when the x-axis of the slope equation is obtained. Its purpose is to show the didactic potential in the study of infinitesimal calculus by using software such as Matlab.

Keywords: *Rolle's Theorem, tangent straight line, means value of the integral.*

1. INTRODUCTION

In this work, a group of exercises of educational interest is performed in order to show facilities that the Matlab software can demonstrate fundamental theorems with numerical analysis. In this group, several functions possessing the characteristic of being well-behaved were selected, such as a trigonometric function combined with an exponential function, in addition to logarithmic, inverse and power functions as well as hyperbolic trigonometric functions. The literature shows a group of similar work such as the follow: Wei-Chi Yang [1] demonstrated how evolving technological tools have led to advances in the teaching and learning of mathematics. He proposed to use the called dynamic geometry software with a computer algebra system; it led to a new way for studying calculus. Another research effort is made by K. A. Bush [2] who presented a useful application of the mean value theorem through the Jensen's inequality that applies to power and logarithmic function. Jingcheng Tong [3] used the theorem of Rolle to introduce a generalization of mean value theorem for integrals. Their generalization involves two functions instead of one and he achieved a very clear geometry explanation. Felix Martinez de la Rosa [4] proposed a route of mean value theorems and simple proofs with geometrical interpretations. According to the author: the mean value theorem is proved based on the application of Rolle's Theorem to "Deus ex machina" function. Also, he mentioned that the full range of results that relate the function with its derivative and integral are called mean value theorem. Likewise, there are some works of Habeebur Maricar et al. [5] made a proof of the theorem is an application of Rolle's Theorem likewise, the article of Abdus Sattar Gazdar [6] in this letter they would further show that since the Rolle's Theorem in a particular case of mean value theorem. Finally, Rovenski [7] was in the opinion that graphing functions with Matlab is much recommended.

Lagrange's Mean Value Theorem assumes that $f \in C[a,b]$ and $f'(x)$ exist for $x \in (a,b)$, and then there is a number c in (a,b) such that [8]

$$f'(c) = \frac{f(b) - f(a)}{b-a} \tag{1}$$

The other fundamental theorem or Barrow's rule says that if f is continuous in $[a,b]$ and F is any primitive of f in $[a,b]$, that is, $F'(x) = f(x)$ then

$$\int_a^b f(x)dx = F(a) - F(b) \tag{2}$$

The geometric interpretation states that there is a point belonging to the interval in which the tangent is parallel to the secant, pursuant to Rolle 's Theorem [8].

The mean value theorem for integrals is based on the fact that if there is a function restricted between two values such that $f \in C[a,b]$, then, there is a number c in (a,b) such that

$$\frac{1}{b-a}\int_a^b f(x)dx = f(c) \tag{3}$$

These theorems make up the cornerstone of calculus results taking you to Cauchy's mean value theorem; likewise, generalizing the mean value theorem takes you to the Taylor expansion and the Maclaurin series. Additionally it leads to the Jacobian and serves for optimization methods such as Lagrange multipliers; in other words, it has tremendous implications.

2. INTEGRAL MEAN VALUE THEOREM

It begins with a trigonometric function

$$f(x)=\frac{4}{5}\sin(x)+\frac{1}{3}\sin(2x) \quad (4)$$

Verify the hypothesis of the theorem in the interval $[0,3]$, a primitive of $f(x)$ is

$$F(x)=-\frac{4}{5}\cos(x)-\frac{1}{6}\cos(2x) \quad (5)$$

The mean value of the function in the interval $[0,3]$ is

$$\frac{1}{3-0}\int_0^3 f(x)dx=\frac{F(3)-F(0)}{3}=\frac{0.6320-(-0.9667)}{3}=0.5329 \quad (6)$$

Indeed, we have that

$$\int_0^3\left(\frac{4}{5}\sin(x)+\frac{1}{3}\sin(2x)\right)dx=1.5986 \quad (7)$$

The area of the rectangle of $b-a=3$ base and $f(c)=0.5329$ height is

$$f(c)*(b-a)\approx 1.5886$$

Figure 1 shows the function and the mean value of the integral.

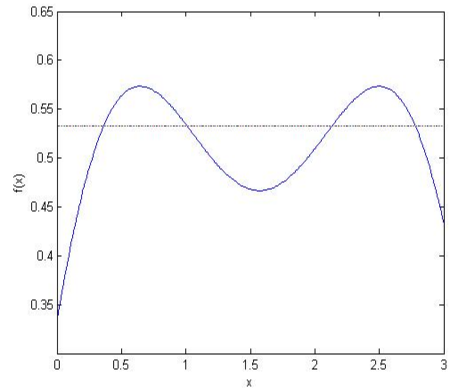

Fig 1: Trigonometric Function

One interesting exercise is the following logarithmic function, which is linked to the history of the theorem of prime numbers

$$\int_1^3 \frac{\ln(x)}{x}dx=0.6035 \quad (6) \quad (8)$$

In addition, the primitive is

$$F(x)=\frac{1}{2}\ln^2(x) \quad (7) \quad (9)$$

Which is determined simply by making

$$u=\ln x \qquad du=\frac{1}{x}dx \quad (10)$$

Now, applying the mean value theorem

$$\frac{1}{3-1}\int_1^3 f(x)dx=\frac{F(3)-F(1)}{2}=\frac{0.6035-0}{2}=0.3018=f(c) \quad (11)$$

Then, the area of the rectangle showed in Figure 2 is

$$f(c)*(b-a)\approx 0.6035$$

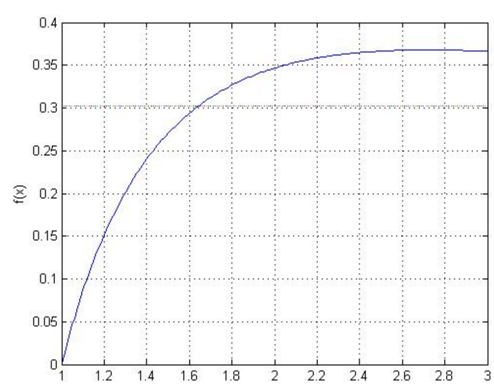

Fig 2: Logarithmic Function

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One example of the inverse function and the hyperbolic trigonometric type, is as follows:

$$\int_0^3 \frac{dx}{\cosh(2x)+1} = 0.49 \tag{12}$$

$$F(x) = \frac{1}{2}\tanh\left(\frac{2x}{2}\right) \tag{13}$$

$$\frac{1}{3-0}\int_0^3 f(x)\,dx = \frac{F(3)-F(0)}{3} = \frac{0.4975-0}{3} = 0.1658 = f(c)\,. \tag{14}$$

Then, the area of the rectangle that is shown in Figure 3 is

$$f(c)*(b-a) \approx 0.4975\,.$$

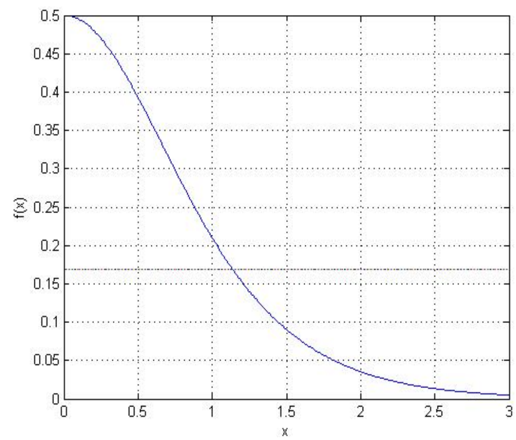

Fig 3: Hyperbolic Trigonometric Function

Obtaining the equation 13 is easy, if we consider that

$$\frac{1}{2}\cosh(2x) + \frac{1}{2} = \cosh^2(x)\ . \tag{15}$$

The following is a different trigonometric function

$$\int_0^5 x\sin^2(x)\,dx = 7.1599 \tag{16}$$

$$F(x) = \frac{x^2}{4} - \frac{x\sin(2x)}{4} - \frac{\cos(2x)}{8} \tag{17}$$

$$\frac{1}{5-0}\int_0^5 f(x)\,dx = \frac{F(5)-F(0)}{5} = \frac{7.0349-(-0.125)}{5} = 1.432 = f(c) \tag{18}$$

Figure 4 shows the function. Then, the area of the rectangle is

$$f(c)*(b-a) \approx 7.159\,.$$

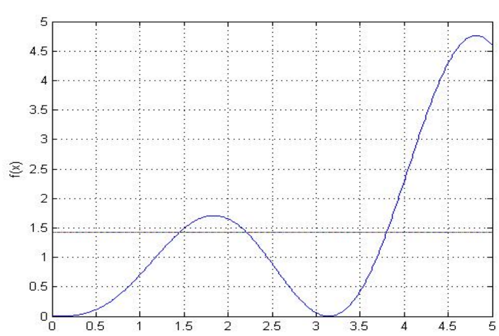

Fig 4: Trigonometric Function

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The following is an example of an exponential function combined to a trigonometric function [9]:

$$\int_{0.5}^{1.5} \cos(x) e^{\sin(x)} dx = 1.09 \quad . \tag{19}$$

Whose primitive is given by:

$$F(x) = e^{\sin(x)} \quad . \tag{20}$$

Applying the theorem we have: (19)

$$\frac{1}{1-0}\int_{0.5}^{1.5} f(x)\,dx = \frac{F(1.5)-F(0.5)}{1} = \frac{2.7115-1.6151}{1} = 1.09 = f(c) \tag{21}$$

(21)

.

Figure 5 shows the function and the rectangle that averages the integral

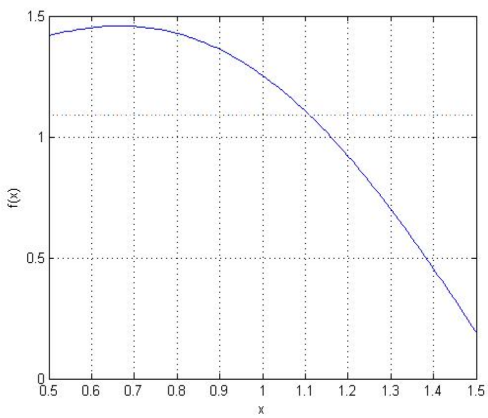

Fig 5: Exponential Function

To determine its primitive $F(x)$, it should be proceeded as follows

$$t = \sin x \qquad dt = \cos x dx \tag{22}$$

$$e^t dt = e^{\sin x} \cos x \, . \tag{23}$$

An example of the power function is the following [9]

$$\int_0^7 \frac{2^x}{\left(1+4^x\right)} dx = 1.1218 \tag{24}$$

.

Whose primitive is given by

$$F(x) = \frac{1}{\log 2} \arctan\left(2^x\right) \tag{25}$$

.

When applying the theorem it results in

$$\frac{1}{7-0}\int_0^7 f(x)\,dx = \frac{F(7)-F(0)}{7} = \frac{2.2549-1.1331}{7} = 0.1603 = f(c) \tag{26}$$

.

Figure 6 shows its behavior and the value that averages it

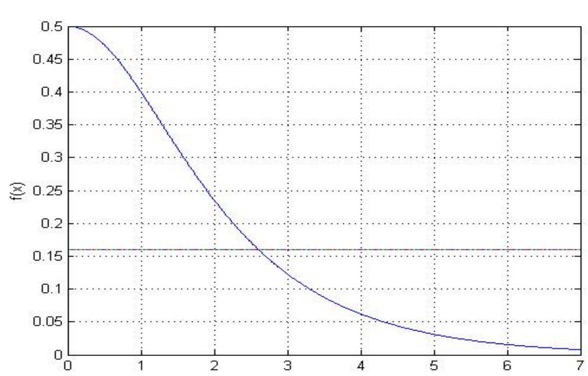

Fig 6: Power Function

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To recognizing $F(x)$, it should be proceeded as follows

$$t = 2^x \qquad dt = 2^x \log 2 dx$$
$$\frac{dt}{\log 2} = 2^x \tag{27}$$

$$\frac{2^x dx}{1+4^x} = \frac{2^x}{1+\left(2^x\right)^2} = \frac{1}{\log 2}\frac{dt}{\left(1+t^2\right)} \tag{28}$$

One example of an algebraic, exponential function is the following [9]

$$\int_0^3 \frac{e^x}{\sqrt{e^{2x}+e^x+1}}dx = 2.5451 \tag{29}$$

Whose primitive is given by

$$F(x) = \log\left|2\sqrt{e^{2x}+e^x+1}+e^{2x}+1\right| \tag{30}$$

When applying the theorem it results in

$$\frac{1}{3-0}\int_0^3 f(x)dx = \frac{F(3)-F(0)}{3} = \frac{4.41113 - -1.8663}{3} = 0.8483 = f(c) \tag{31}$$

Figure 7 shows its average value .

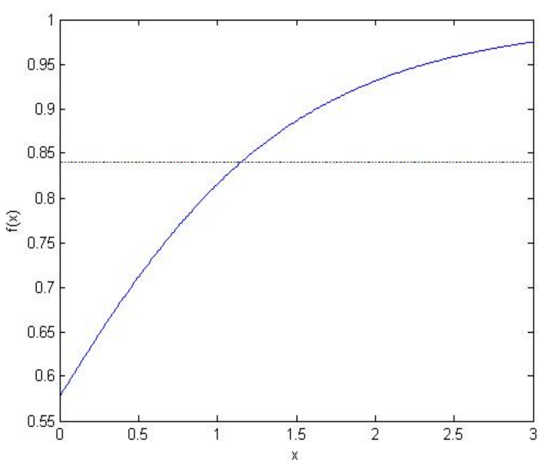

Fig 7: Algebraic Exponential Function

To determine $F(x)$, it should be proceeded as follows [9]

First change of variable

$$t = e^x \qquad dt = e^x dx$$
$$\frac{e^x}{\sqrt{e^{2x}+e^x+1}} = \frac{dt}{\sqrt{t^2+t+1}} \tag{32}$$

Second change of variable

$$y = t + \frac{1}{2} \qquad dy = dt$$
$$\frac{dt}{\sqrt{t^2+t+1}} = \frac{dy}{\sqrt{y^2+\frac{3}{4}}},$$

Third change of variable

$$y = \frac{\sqrt{3}}{2}\tan(z) \qquad dy = \frac{\sqrt{3}}{2\cos^2(z)}dz \tag{33}$$

$$\frac{dy}{\sqrt{y^2+\frac{3}{4}}} = \frac{dz}{\cos(z)} \tag{34}$$

Allowing to find the primitive $F(x)$.

All these functions are well-behaved and represent a good form to apply the fundamental calculus theorems; this is what they have in common. Next, the same functions are used to find the equation of the tangent straight line.

3. AGRANGE'S THEOREM

Next, some straight line tangents generated are presented; first, we begin with the trigonometric function and its respective derivative function

$$f(x) = x\sin^2(x)$$
$$f'(x) = \sin^2(x) + 2x\cos(x)\sin(x) \tag{35}$$

For an interval between $[3.5, 4.5]$ we have

$$f'(c) = \frac{4.3 - 0.43}{4.5 - 3.5} = 3.87 \quad (36)$$

Therefore, the tangent straight line is

$$y = 3.87x - 13.28 \quad (37)$$

Figure 8 shows the function in this interval and its tangent straight line

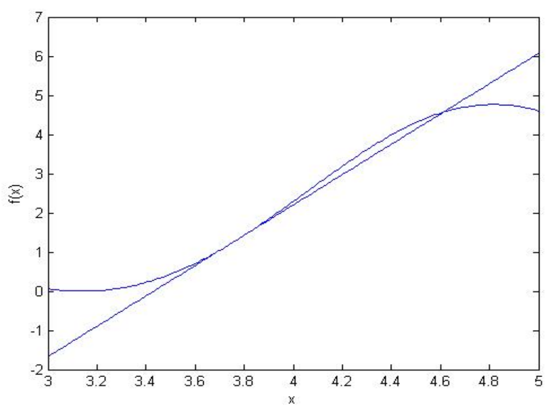

Fig 8: Trigonometric Function

Another case of interest is the logarithmic function, an example of which is found next

$$f(x) = \frac{\ln x}{x} \qquad f'(x) = \frac{1 - \ln x}{x^2} \quad (38)$$

Applying the theorem in the interval $[1.5, 2.5]$ we have

$$f'(c) = \frac{0.3665 - 0.2703}{2.5 - 1.5} = 0.0962 \quad (39)$$

Whose tangent is

$$y = 0.0962x + 0.1551 \quad (40)$$

Figure 9 shows the relation between the tangent straight line and the function

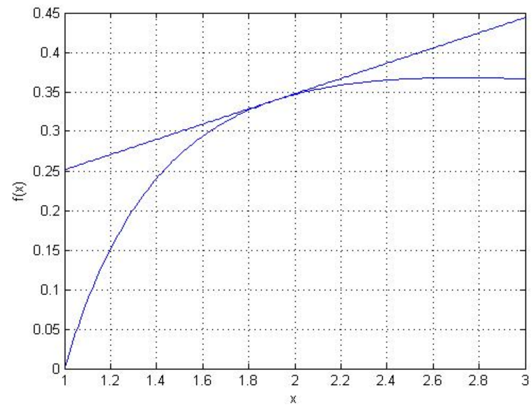

Fig 9: Logarithmic Function

The following hyperbolic function in the interval $[0.5, 2.5]$ and its respective derivative function is

$$f(x) = \frac{1}{\cosh(2x) + 1}$$
$$f'(x) = \frac{2\sinh(2x)}{\left(\cosh(2x) + 1\right)^2} \quad (41)$$

Applying the theorem we have

$$f'(c) = \frac{0.0904 - 0..3932}{1.5 - 0.5} = -0.3032 \quad (42)$$

Therefore, the tangent straight line is

$$y = 0.5125 - 0.03032x \quad (43)$$

Figure 10 shows the tangent straight line and hyperbolic trigonometric function

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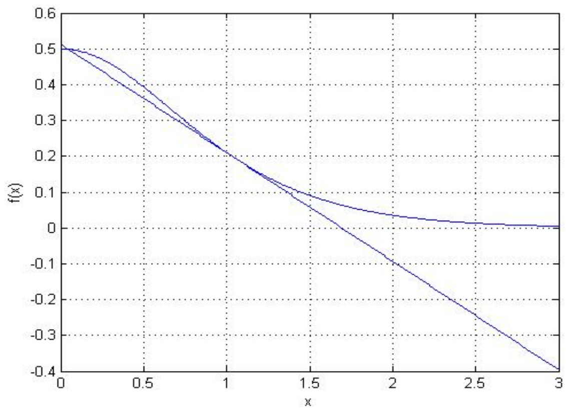

Fig 10: Hyperbolic Trigonometric Function

For an exponential function and its derivative in the interval $\left[0.7,1.3\right]$

$$f\left(x\right)=\cos\left(x\right)e^{\sin\left(x\right)} \qquad f'\left(x\right)=e^{\sin\left(x\right)}\left[\cos^2\left(x\right)+\sin\left(\right.\right. \tag{44}$$

Applying the theorem we have

$$f'\left(c\right)=\frac{0.7011-1.456}{1.3-0.7}=-1.2582 \tag{45}$$

Therefore, the tangent straight line is

$$y=2.5116-1.2582x \tag{46}$$

Figure 11 illustrates the equation and function of this straight line

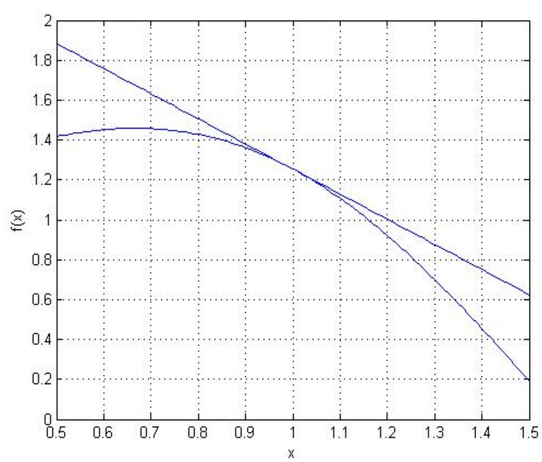

Fig 11: Exponential Function

Finally, we have the following power function in the interval $\left[1.2,2.2\right]$

$$f\left(x\right)=\frac{2^x}{\left(1+4^x\right)} \qquad f'\left(x\right)=\frac{2^x\log 2}{\left(1+4^x\right)}-\frac{2^x\log 4}{\left(1+4^x\right)^2} \tag{47}$$

Applying the theorem we have

$$f'\left(c\right)=\frac{0.2078-0.3659}{2.2-1.2}=-0.1581 \tag{48}$$

Thus, we obtained the tangent equation

$$y=0.549-0.1581x \tag{49}$$

Graph 12 shows the function and its respective straight line

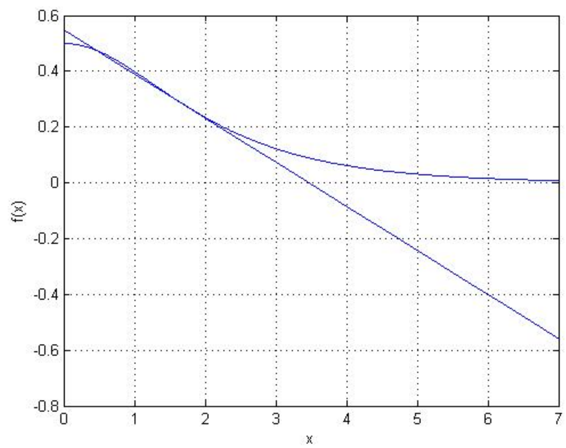

Fig 12: Power Function

4. DISCUSSION OF RESULTS

This paper traces how to obtain the analytical solution of each integral of the equations (10), (15) (23),(28) and (34). Similarly, tracing the secant line of the functions in the corresponding interval in Figures 7 to 11 is also relatively easy. The calculation of the root solutions for each generated occasion by equating the derivative function to $f'\left(c\right)=0$ was avoided. Equation (8) is interesting because it is connected to the theorem of prime numbers. The challenge is passing to two dimensions by using this kind of functions since this work is limited to only one dimension; besides of including the Cauchi Theorem.

5. CONCLUSION

We applied Lagrange's Theorem on a basic group of functions. We verified that the algebraic solution is introduced in the graph solution, and we used the software in order to determine the roots that are needed when equalizing the x-axis to the derivative function. We were helped to understand an important chain of theorems such as the Rolle's Theorem and the Lagrange theorem and it can be easily extended to the Maclaurin series, Cauchy theorem, Jacobian functions, and other fundamental pieces of the infinitesimal calculus along with the mathematical development for each problem. This work demonstrates a form of redesigning modern mathematical courses. What follows is the same method to investigate the connection with number theory.

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