

# General Upper Limit of the Age of the Universe

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## ABSTRACT

The aim of the study is to describe the general upper limit of the age of the universe with the help of different cosmological models of the universe. Here homogeneous and isotropic assumptions of the observed universe are not strictly followed to calculate the present age of the universe. Einstein equations play an important role in cosmology to determine the present age of the universe. The study stresses on the works of Friedmann, Robertson-Walker (FRW) universe and Raychaudhuri equations. The paper tries to find the general upper limit of the age of the universe with easier mathematical calculations.

**Keywords:** Einstein equation, geodesic, Hubble constant, space-time manifold, universe.

## 1. INTRODUCTION

Friedmann, Robertson-Walker (FRW) universe is based on the assumption that the universe is exactly homogeneous and isotropic. In FRW models there is an all encompassing big bang singularity in the past as the origin from which the universe emerges in a very hot phase and continues its expansion as it cools. The three parameters  $H_0$ ,  $q_0$  and  $\rho_0$  fully characterize these models, which are called the Hubble constant, the deceleration parameter and the density parameter, respectively. The Hubble constant is believed to be in the range of  $40-120\text{km}^{-1}\text{Mpc}^{-1}$ . There is a considerable uncertainty in determining the value of deceleration parameter and as a result, the question of the present density of the universe; or whether it is open or closed model, remains uncertain. The determination of present age and density of the universe are two very important issues in cosmology, as they determine the future evolution and the nature of the universe.

The paper is organized as follows:

In section-2 we briefly discuss the portion of general relativity related to this paper following [1, 6, 7]. FRW models are written in section-3 for the convenience to discuss the paper clearly following Islam [3]. Raychaudhuri equations are introduced in section-4 [6, 9]. The upper limit of the age of the universe, the main portion of the paper, is described in section-5 and the final section is of the concluding remarks.

## 2. STUDY RELATED DISCUSSION OF GENERAL RELATIVITY

General relativity models the physical universe as a four-dimensional  $C^\infty$  Hausdorff differentiable space-time manifold  $M$  with a Lorentzian metric  $g$  of signature  $(-,+,+,+)$  which is topologically connected, paracompact, space-time orientable and without boundary. It was realized that though locally the laws of physics are those of special relativity and space-time is very nearly flat rise to a non-flat, curved continuum which would also admit a suitable differential structure. The

universe is not simply a random collection of irregular distributed matter, but it is a single entity, all parts of which are connected. When considering the large scale structure of the universe, the basic constituents are galaxies, which are congregation of more than  $10^{10}$  stars bound together by their mutual gravitational attraction. The universe is the totality of galaxies which are causally connected [1].

### 2.1 Some Related Definitions

Now we will discuss some definitions related to this paper. The definitions will help the readers to understand the concept of the paper easily.

#### Topological Space

Let  $M$  be a non-empty set. A class  $T$  of subsets of  $M$  is a topology on  $M$  if  $T$  satisfies the following three axioms:

1.  $M$  and  $\phi$  belong to  $T$ ,
2. the union of any number of open sets in  $T$  belongs to  $T$ , and
3. the intersection of any two sets in  $T$  belongs to  $T$ .

The members of  $T$  are open sets and the space  $(M, T)$  is called topological space.

#### Connected Sets

A subset  $A$  of a topological space  $M$  is disconnected if  $\exists$  open subsets  $G$  and  $H$  of  $M$  such that  $A \cap G$  and  $A \cap H$  are disjoint non-empty sets whose union is  $A$ . In this case  $G \cup H$  is called a disconnection of  $A$ . A set is connected if it is not disconnected.

#### Hausdorff Space

A topological space  $M$  is a Hausdorff space if for pair of distinct points  $p, q \in M$  there are disjoint open sets  $U_\alpha$  and  $U_\beta$  in  $M$  such that  $p \in U_\alpha$  and  $q \in U_\beta$ .

#### Paracompact Space

An atlas  $\{U_\alpha, \phi_\alpha\}$  is called locally finite if there is an open set containing every  $p \in M$  which intersects

only a finite number of the sets  $U_\alpha$ . A manifold  $M$  is called a paracompact if for every atlas there is locally finite atlas  $\{O_\beta, \mathcal{V}_\beta\}$  with each  $O_\beta$  contained in some  $U_\alpha$ . Let  $V^\mu$  be a time like vector, then paracompactness of manifold  $M$  implies that there is a smooth positive definite Riemann metric  $K_{\mu\nu}$  (discussed later) defined on  $M$ .

### Space-time Manifold

The above properties of general relativity are suitable when we consider for local physics. As soon as we investigate global features then we face various pathological difficulties such as, the violation of time orientation, possible non-Hausdorff or non-paracompactness, disconnected components of space-time etc. Such pathologies are to be ruled out by means of reasonable topological assumptions only. However, we like to ensure that the space-time is causally well-behaved. We will consider the space-time Manifold  $(M, g)$  which has no boundary. By the word 'boundary' we mean the 'edge' of the universe which is not detected by any astronomical observations. It is common to have manifolds without boundary; for example, for two-spheres  $S^2$  in  $\mathcal{R}^3$  no point in  $S^2$  is a boundary point in the induced topology on the same implied by the natural topology on  $\mathcal{R}^3$ . All the neighborhoods of any  $p \in S^2$  will be contained within  $S^2$  in this induced topology. We shall assume  $M$  to be connected i.e., one cannot have  $M = X \cup Y$ , where  $X$  and  $Y$  are two open sets such that  $X \cap Y \neq \emptyset$ . This is because disconnected components of the universe cannot interact by means of any signal and the observations are confined to the connected component wherein the observer is situated. It is not known if  $M$  is simply connected or multiply connected.  $M$  is assumed to be Hausdorff, which ensures the uniqueness of limits of convergent sequences and incorporates our intuitive notion of distinct space-time events [6].

### Space-time Orientation

Let  $B$  be the set of all ordered basis  $\{e_i\}$  for vector space  $T_p$ , the tangent space at point  $p$ . If  $\{e_i\}$  and  $\{e_j\}$  are in  $B$ , then we have  $e_j = a_j^i e_i$ . If we denote the matrix  $(a_{ij})$  then  $\det(a) \neq 0$ . An  $n$ -dimensional manifold  $M$  is called orientable if  $M$  admits an atlas  $\{U_i, \phi_i\}$  such that whenever  $U_i \cap U_j \neq \emptyset$

then the Jacobian,  $J = \det\left(\frac{\partial x^i}{\partial x^j}\right) > 0$ , where  $\{x^i\}$  and

$\{x^j\}$  are local coordinates in  $U_i$  and  $U_j$  respectively.

The Möbius strip is a non-orientable manifold. A vector defined at a point Möbius strip with a positive orientation comes back with a reversed orientation in

negative direction when it traverses along the strip to come back to the same point [4, 6].

### Hypersurface

The Minkowski space-time is defined by;

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2. \quad (1)$$

Here the surface  $t = 0$  is a three-dimensional surface with the time direction always normal to it. Any other surface  $t = \text{constant}$  is also a spacelike surface in this sense. Let  $S$  be an  $(n-1)$ -dimensional manifold. If there exists a  $C^\infty$  map  $\phi: S \rightarrow M$  which is locally one-one i.e., if there is a neighborhood  $N$  for every  $p \in S$  such that  $\phi$  restricted to  $N$  is one-one, and  $\phi^{-1}$  is a  $C^\infty$  as defined on  $\phi(N)$ , then  $\phi(S)$  is called an embedded sub-manifold of  $M$ . A hyper surface  $S$  of any  $n$ -dimensional manifold  $M$  is defined as an  $(n-1)$ -dimensional embedded sub-manifold of  $M$ . Let  $V_p$  be the  $(n-1)$ -dimensional subspace of  $T_p$  of the vectors tangent to  $S$  at any  $p \in S$  from which follows that there exists a unique vector  $n^a \in T_p$  and is orthogonal to all the vectors in  $V_p$ .  $n^a$  is called the normal to  $S$  at  $p$ . If the magnitude of  $n^a$  is either positive or negative at all points of  $S$  without changing the sign, then  $n^a$  could be normalized so that  $g_{ab} n^a n^b = \pm 1$  where  $g_{ab}$  is an indefinite metric in the sense that the magnitude of non-zero vector could be either positive, negative or zero and is defined by;

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu. \quad (2)$$

If  $g_{ab} n^a n^b = -1$  then the normal vector is time like everywhere and  $S$  is called a spacelike hyper surface. If the normal is spacelike everywhere on  $S$  with a positive magnitude,  $S$  is called a time like hyper surface. Finally,  $S$  is null hyper surface if the normal  $n^a$  is null at  $S$  [4, 6].

### 2.2 Einstein Field Equations

Einstein's field equations are very important in general relativity, cosmology and particle physics.

We can write a relation of general relativity as;

$$A_{\mu;\nu;\sigma} - A_{\mu;\sigma;\nu} = R_{\mu\nu\sigma}^\alpha A_\alpha, \quad (3)$$

where the tensor;

$$R_{\mu\nu\sigma}^\alpha = \Gamma_{\mu\sigma;\nu}^\alpha - \Gamma_{\mu\nu;\sigma}^\alpha + \Gamma_{\beta\nu}^\alpha \Gamma_{\mu\sigma}^\beta - \Gamma_{\beta\sigma}^\alpha \Gamma_{\mu\nu}^\beta \quad (4)$$

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is a tensor of rank four and called Riemann curvature tensor. Taking inner product of both sides of (4) with  $g_{\rho\alpha}$  one gets covariant curvature tensor

$$R_{\rho\mu\nu\sigma} = \frac{1}{2} \left( \frac{\partial^2 g_{\rho\sigma}}{\partial x^\nu \partial x^\mu} + \frac{\partial^2 g_{\mu\nu}}{\partial x^\sigma \partial x^\rho} - \frac{\partial^2 g_{\mu\sigma}}{\partial x^\nu \partial x^\rho} - \frac{\partial^2 g_{\rho\nu}}{\partial x^\sigma \partial x^\mu} \right) + g_{\alpha\lambda} \left( \Gamma_{\mu\nu}^\alpha \Gamma_{\rho\sigma}^\lambda - \Gamma_{\mu\sigma}^\alpha \Gamma_{\rho\nu}^\lambda \right). \quad (5)$$

Contraction of curvature tensor (5) gives Ricci tensor;

$$R_{\mu\nu} = g^{\lambda\sigma} R_{\lambda\mu\sigma\nu}. \quad (6)$$

Further contraction of (6) gives Ricci tensor;

$$\hat{R} = g^{\lambda\sigma} R_{\lambda\sigma}. \quad (7)$$

The space-time  $(M, g)$  is said to have a flat connection iff;

$$R_{\nu\sigma}^\mu = 0. \quad (8)$$

This is necessary and sufficient condition for a vector at a point  $p$  to remain unaltered after parallel transported along an arbitrary closed curve through  $p$ . This is because all such curves can be shrunk to zero, in which case the space-time is simply connected.

Let us assume the matter content of the universe as a perfect fluid. The energy momentum tensor  $T^{\mu\nu}$  is defined as;

$$T^{\mu\nu} = \rho_0 u^\mu u^\nu \quad (9)$$

where  $\rho_0$  is the proper density of matter, and if there is no pressure, and  $u^\mu = X^\mu = \frac{dx^\mu}{dt}$  is a tangent vector. A perfect fluid is characterized by pressure  $p = p(x^\mu)$ , then the energy momentum tensor can be written as;

$$T^{\mu\nu} = (\rho + p) u^\mu u^\nu + p g^{\mu\nu}, \quad (10)$$

where  $\rho$  is the scalar density of matter.

The principle of local conservation of energy and momentum states that;

$$T^{\mu\nu}_{;\nu} = 0. \quad (11)$$

In the Einstein field equations Einstein stated that the universe is static. Einstein introduced a

cosmological constant  $\Lambda (\approx 0)$  for static universe solutions as;

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = -\frac{8\pi G}{c^4} T_{\mu\nu}. \quad (12)$$

Einstein's field equations (12) for  $\Lambda = 0$  can be written as;

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -\frac{8\pi G}{c^4} T_{\mu\nu}. \quad (13)$$

where  $G = 6.673 \times 10^{-11}$  is the gravitational constant and  $c = 10^8$  m/s is the velocity of light but in relativistic unit  $G = c = 1$ . Hence in relativistic units (13) becomes;

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -8\pi T_{\mu\nu}. \quad (14)$$

It is clear that divergence of both sides of (13) and (14) is zero. For empty space  $T_{\mu\nu} = 0$  then  $R_{\mu\nu} = \Lambda g_{\mu\nu}$ , then;

$$R_{\mu\nu} = 0 \text{ for } \Lambda = 0, \quad (15)$$

which is Einstein's law of gravitation for empty space.

The Schwarzschild exterior solution of Einstein's field equations describes the gravitational fields outside a spherically symmetric star where there is no matter present and space-time is empty. The space-time distance  $ds$  for gravitating mass  $m$  in  $(t, r, \theta, \phi)$  coordinates between two infinitesimally separated events is given by the metric;

$$ds^2 = -\left(1 - \frac{2m}{r}\right) dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \quad (16)$$

The above space-time has an apparent singularity at  $r = 2m$  as seen by the divergence of metric components of equation (16). It was thought initially that the above represents a singularity in the space-time itself and that physics goes seriously wrong at  $r = 2m$ . After some efforts it is realized that this is not a genuine space-time singularity but merely a coordinate defect, and what was really needed was an extension of the Schwarzschild manifold. Such an extension of the space-time was obtained by Kruskal and Szekeres [5, 10] and this may be regarded as an important insight involving a global approach. However at  $r = 0$  there is a genuine curvature singularity where the Kretschman scalar  $\alpha = R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} \rightarrow \infty$  i.e., space-time curvature components tend to infinity [6].

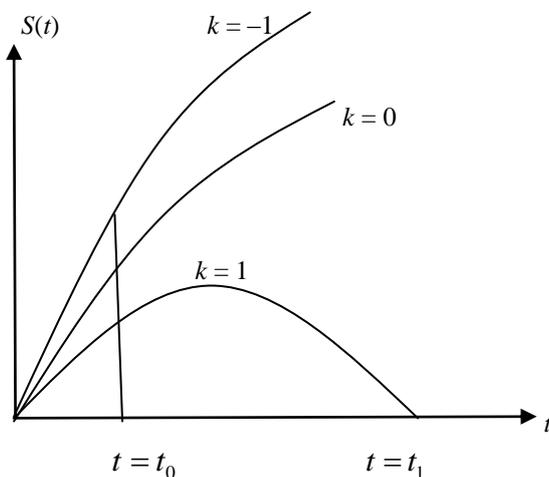
### 3. FRIEDMANN, ROBERTSON-WALKER (FRW) MODELS

Soon after the Einstein's field equations were discovered, Friedmann showed that the universe must have originated a finite time ago from an epoch of infinite density and curvatures where all the known physical laws break down which we call 'big bang'. So we cannot predict what was happened in the period of big bang and before big bang. Therefore, we can say that it is a singularity in space-time topology [3].

In  $(t, r, \theta, \phi)$  coordinates the Robertson-Walker line element is given by;

$$ds^2 = -dt^2 + S^2(t) \left[ \frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2 \theta d\phi^2) \right] \quad (17)$$

where  $S(t)$  is the scale factor and  $k$  is a constant which denotes the spatial curvature of the three-space and could be normalized to the values  $+1, 0, -1$ . When  $k = 0$  the three-space is flat and the model is called Einstein de-Sitter static model, when  $k = +1$  and  $k = -1$  the three-space are of positive and negative constant curvature; these incorporate the closed and open Friedmann models respectively (Figure 1).



**Fig 1:** The behavior of the curve  $S(t)$  for the three values  $k = -1, 0, +1$ ; the time  $t = t_0$  is the present time and  $t = t_1$  is the time when  $S(t)$  reaches zero again for  $k = +1$ .

### 4. RAYCHAUDHURI EQUATIONS

Raychaudhuri equations play important roles to describe the gravitational focusing and space-time singularities. Amal Kumar Raychaudhuri established it in 1955 to describe gravitational focusing properties in cosmology. Raychaudhuri [9] developed a set of equations which contribute to find the gravitational

focusing in the space-time. Raychaudhuri equations [9] can be written as;

$$\frac{d\theta}{dt} = -R_{\mu\nu} V^\mu V^\nu - \frac{1}{n} \theta^2 - 2\sigma^2 + 2\omega^2. \quad (18)$$

( $n = 2$  for null geodesic, and  $n = 3$  for time like geodesics), where  $V^\mu$  denotes the time like tangent vector in the manifold to the congruence. Raychaudhuri equations (18) which describe the rate of change of the volume expansion as one moves along the time like geodesic curves in the congruence [6].

Here  $\theta > 0$  is expansion,  $\sigma > 0$  is shear and  $\omega$  is rotation/vorticity tensors which are defined as follows:

$$\theta_{\mu\nu} = V_{(\alpha;\beta)} h_\mu^\alpha h_\nu^\beta$$

$$\sigma_{\mu\nu} = \theta_{\mu\nu} - \frac{1}{3} h_{\mu\nu} \theta$$

$$\omega_{\mu\nu} = h_\mu^\alpha h_\nu^\beta V_{[\alpha;\beta]}.$$

The spatial part  $h_{\mu\nu}$  of the metric tensor is defined as;

$$h_{\mu\nu} = g_{\mu\nu} + V_\mu V_\nu. \quad (19)$$

### 5. UPPER LIMIT OF THE AGE OF THE UNIVERSE

#### 5.1 Matters of the Universe

The development in modern astrophysics states that dark matter surrounds the bright stars and galaxies, and constitutes the dominant material content of our universe. The dark matter seems to be present on all distance scales from the local neighborhood of our sun and the Milky Way, to clusters and super clusters of galaxies and also up to the expansion scale of the universe (about 14 billion light years). In local scale the expansion of dark matter can be had in terms of brown dwarfs but on larger scales no satisfactory explanation exists in terms of ordinary matter. At present time, the evidence of the dark matter seems to be (i) low mass, faint stars (ii) massive black holes and (iii) massive neutrinos, axions or particles predicted by super symmetry. Of these axions are the most popular candidate as it seems to fit best the astrophysical requirements. Axions could have been produced in the early universe, and if the axion has a small, non-zero rest mass, and then they would be gravitationally dominant today. The existence of axions was originally invoked when Peccei and Quinn [8] explained the property of C-P (charge and parity) conservation of strong interactions.

**5.2 Mathematical Approach**

In FRW models the universe is homogeneous and isotropic but there is no fundamental physical justification that isotropy and homogeneity are strictly obeyed in all regions of space and at all epochs of time. Here we consider the universe is inhomogeneous and anisotropic by means of a general globally hyperbolic space-time. The standard cosmological space-times such as FRW models, Bianchi or steady state cosmologies, are all globally hyperbolic. Let  $S$  be a spacelike Cauchy surface. The initial data specified on such a hyper surface can be evolved into future or past to predict the future and past state of the universe by means of hyperbolic differential equation. We consider one parameter family  $(S_t)$  of spacelike Cauchy surfaces which make it possible to refer to the state of the universe at any epoch of a point conjugate to a spacelike hyper surface  $S_t$  along a time like geodesic  $\gamma$ .

In this paper we would like to investigate within the globally hyperbolic framework the extension into the past of time like geodesic trajectories by considering the gravitational focusing effect of the matter in a space-time. Let the present epoch be characterized by a spacelike global Cauchy surface  $S_0$ , at  $t = 0$  and the matter distribution is given by;

$$\left( T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right) V^\mu V^\nu \geq 0 \tag{20}$$

where  $V^\mu$  is unit time like vector and  $T_{\mu\nu}$  is stress-energy tensor. The term  $T_{\mu\nu} V^\mu V^\nu$  is the energy density measured by a time like observer with the unit tangent four velocity of the observer,  $V^\mu$ . Relation (20) is called strong energy condition, which states that there are no negative energy fields in the space-time. By Einstein equations (13), equation (20) implies;

$$R_{\mu\nu} V^\mu V^\nu \geq 0. \tag{21}$$

In our study we do not need the exact homogeneity or isotropy of mass-energy distribution on  $S_0$ , we assume for the simplicity of consideration that there exists a minimum for energy distribution on  $S_0$ . By the observed expansion of the universe, this should exhibit a non-decreasing behavior in the past, which means that there exist some  $B > 0$  such that;

$$R_{\mu\nu} V^\mu V^\nu \geq B > 0. \tag{22}$$

at present and all past epochs.

For  $\omega = 0$  (18) becomes;

$$\frac{d\theta}{dt} = -R_{\mu\nu} V^\mu V^\nu - \frac{1}{3} \theta^2 - 2\sigma^2. \tag{23}$$

A time like geodesic  $\gamma(t)$  will be orthogonal to  $S_0$  provided the expansion  $\theta$  along  $\gamma(t)$  satisfies  $\theta = \chi^i_j$  at  $S_0$ , where  $\chi_{ij}$  is the second fundamental form of the spacelike hyper surface. Let  $\theta \equiv \frac{1}{z} \frac{dz}{dt}$  with  $z = x^3$  then (23) becomes [3];

$$\frac{d^2 x}{ds^2} + H(t)x = 0 \tag{24}$$

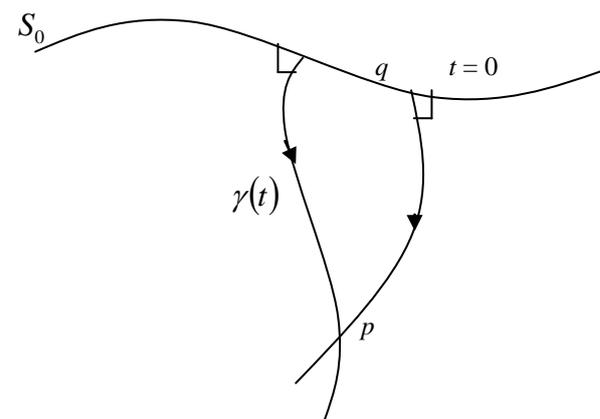
where  $H(t) = \frac{1}{3} (R_{\mu\nu} V^\mu V^\nu + 2\sigma^2)$ .

Now we have to find a point  $p$  conjugate to  $S_0$  along  $\gamma(t)$ , that is to find a solution  $x(t)$  to equation (24) which vanishes at  $p$  (Figure 2). So that initially for some constant  $\alpha$ ;

$$x(0) = \alpha, \text{ and}$$

$$\frac{dx}{dt} = \frac{1}{3} \theta x, \quad \left. \frac{dx}{dt} \right|_{t=0} = \frac{1}{3} \alpha \theta = \alpha \chi^\mu_\mu \tag{25}$$

which vanishes at  $p$ .



**Fig 2:** Point  $p$  is conjugate to  $S_0$  along  $\gamma(t)$ .

To solve equation (24) we use Sturm comparison theorem which compares the distribution zeros of the solutions  $u(t)$  and  $v(t)$  of the equations [2] ;

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$$\frac{d^2u}{dt^2} + G_1(t)u = 0, \quad (26)$$

$$\frac{d^2v}{dt^2} + G_2(t)v = 0$$

where  $G_1 \leq G_2$  in an interval  $(a, b)$ . The theorem then shows that if  $u(t)$  has  $m$  zeros in  $a < t < b$  then  $v(t)$  has at least  $m$  zeros in the same interval and the  $k^{\text{th}}$  zero of  $v(t)$  is must be earlier than the  $k^{\text{th}}$  zero of  $u(t)$ .

Now let,

$$A^2 = \min H(t) = \min \frac{1}{3} (R_{\mu\nu} V^\mu V^\nu + 2\sigma^2)$$

and consider the equation;

$$\frac{d^2x}{dt^2} + A^2x = 0. \quad (27)$$

If we apply the Sturm theorem to equations (24) and (27) we observe that if the solution to equation (27) satisfying the initial conditions (25) has a zero in the interval  $0 < t < t_1$ , then the solution of equation (24) defined by the same initial conditions must have a zero in the same interval, which must occur before the zero of the solution of equation (27). Now the general solution of (27) can be written as;

$$x = C_1 \sin(C_2 + At). \quad (28)$$

Let us choose the initial condition as;

$$x(0) = \frac{1}{\left( (\chi_\mu^\mu)^2 + A^2 \right)^{1/2}},$$

$$\left. \frac{dx}{dt} \right|_{t=0} = \frac{\chi_\mu^\mu}{\left( (\chi_\mu^\mu)^2 + A^2 \right)^{1/2}}. \quad (29)$$

Since the universe is expanding everywhere so  $\chi_\mu^\mu < 0$  on  $S_0$ . The universe may contract or it may expand at some places and may contract in some other places, but we shall not consider such possibilities here, instead we consider only expanding behavior. Using initial conditions (29), solution (28) can be written as;

$$x = \frac{1}{A} \sin(\theta - At) \quad (30)$$

with

$$\theta = \sin^{-1} \left\{ \frac{A}{\left( (\chi_\mu^\mu)^2 + A^2 \right)^{1/2}} \right\}.$$

We have  $0 < \theta < \frac{\pi}{2}$  and a zero for  $x$  must occur within the interval;

$$0 \leq At \leq \frac{\pi}{2} \Rightarrow 0 \leq t \leq \frac{\pi}{2A}, \quad (31)$$

i.e., if  $\gamma(t)$  is any time like curve geodesic orthogonal to  $S_0$ , then there must be a point  $p$  on  $\gamma(t)$ , conjugate to  $S_0$ , within the above interval where  $A^2 = \min H(t) > 0$ .

No time like curve from  $S_0$  can be extended into the past beyond the proper time length  $\frac{\pi}{2A}$ . Let  $q$  be an event on  $S_0$  and  $\gamma$  be a past directed, endless time like curve from  $q = \gamma(0)$ . Let  $\gamma$  can be extended to arbitrary values of proper time in the past, then choose  $p = \gamma\left(\frac{\pi}{2A}\right)$  to be an event on this trajectory. Then there exists a time like  $\gamma'$  from  $p$  orthogonal to  $S_0$  along which the proper time lengths of all non-spacelike curves from  $p$  to  $S_0$  are maximized and further,  $\gamma'$  does not contain any conjugate point to  $S_0$  between  $p$  and  $S_0$ . Again, we have shown that any time like geodesic  $\gamma(t)$  must contain a point conjugate to  $S_0$  within the proper time length  $\frac{\pi}{2A}$ . But this is impossible and we can say that time like curve from can be extended into the past beyond the proper time length  $\frac{\pi}{2A}$  i.e.,  $t_{\max} = \frac{\pi}{2A}$ .

Now the above results can be applied to obtain general upper bounds to the age of a globally hyperbolic universe in the following manner.

By using (10) and (13) we can write,

$$R_{\mu\nu} V^\mu V^\nu = 8\pi G \left( T_{\mu\nu} V^\mu V^\nu + \frac{1}{2} T \right)$$

$$R_{\mu\nu} V^\mu V^\nu = 8\pi G (\rho + P) (V^4)^2 - 4\pi G \rho + 4\pi GP$$

$$\Rightarrow R_{\mu\nu} V^\mu V^\nu \geq 4\pi G(\rho + 3P). \quad (32)$$

### 4.3 Limit of the Age of the Universe

We assume that energy density of the present universe is mainly contributed by the non-relativistic free gas of neutrinos for which  $P \ll \rho$  then (32) becomes;

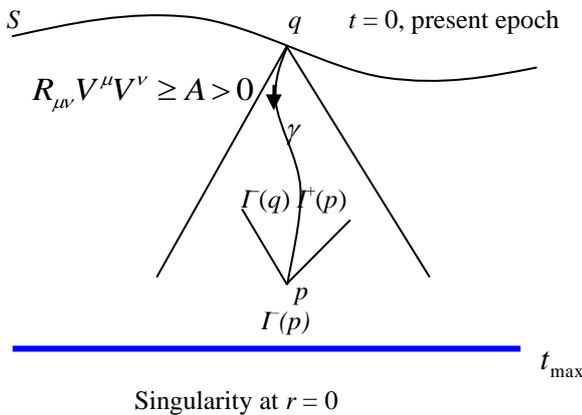
$$R_{\mu\nu} V^\mu V^\nu \geq 4\pi G\rho$$

$$A^2 = \min \frac{1}{3} (R_{\mu\nu} V^\mu V^\nu + 2\sigma^2) \geq \frac{4}{3} \pi \rho G$$

where  $\rho$  is the present density of the universe. Hence the maximum possible age of the universe  $t_{\max}$ , is given by;

$$t_{\max} = \frac{\pi}{2A} = \frac{\pi}{2} \left( \frac{3}{4\pi\rho G} \right)^{1/2} = \pi \left( \frac{3}{16\pi\rho G} \right)^{1/2} \quad (33)$$

with the basis of general globally hyperbolic space-time (Figure 3).



**Fig 3:** No time like curve  $\gamma$  from the surface  $S$  extends the maximum limit  $t_{\max}$  in the past and must encounter a space-time singularity before this epoch.

In radiation dominated models we can write

$$P = \frac{1}{3} \rho, \text{ then (32) becomes;}$$

$$R_{\mu\nu} V^\mu V^\nu \geq 8\pi\rho G. \quad (34)$$

Then (33) becomes;

$$t_{\max} = \pi \left( \frac{3}{32\pi\rho G} \right)^{1/2}. \quad (35)$$

Both (33) and (35) give upper limits to the age of the universe irrespective of whether or not the distribution of whether on  $S_0$  is isotropic and homogeneous which does not assume.

The average mass density as indicated by the visible galaxies is about  $10^{-30} \text{ gm cm}^{-3}$ . The X-ray observations strongly favor the existence of a hot ionized intergalactic gas within the cluster of galaxies whereas weakly interacting massive neutrinos could be another source. If the microwave background radiation (MBR) as having some kind of global origin, then  $\rho_{MBR}$  provides a firm lower limit of the  $\rho_{\min}$  sought for and general upper limit to the age of the universe as given by (35) is;

$$t_{\max} = \pi \left( \frac{3}{32\pi\rho G} \right)^{1/2} = 3.2 \times 10^{12} \text{ years}, \quad (36)$$

$$\rho_{MBR} = 4.4 \times 10^{-3} \text{ gm cm}^{-3}.$$

The relationships (34) and (36) provide upper limits to the age even when allowing for departures from homogeneity and isotropy. If we take the contribution by matter into account, we have to choose an entire range of densities as suggested by the above mentioned possibilities.

The average matter density arising from all possible sources is believed to be between  $10^{-31}$  to  $10^{-28} \text{ gm cm}^{-3}$ .

For  $\rho = 10^{-31} \text{ gm cm}^{-3}$  in (36) we find;

$$t_{\max} = \pi \left( \frac{3}{32\pi\rho G} \right)^{1/2} = 9.43 \times 10^{10} \text{ years}, \quad (37)$$

and for  $\rho = 10^{-28} \text{ gm cm}^{-3}$  in (36) we find;

$$t_{\max} = \pi \left( \frac{3}{32\pi\rho G} \right)^{1/2} = 0.94 \times 10^{10} \text{ years}. \quad (38)$$

We observe that the general upper bound on the age of the universe varies from about  $0.94 \times 10^{10}$  years for the highest possible present density, to about  $9.43 \times 10^{10}$  years for the lowest possible value of the density.

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The following Table 1 shows the possible maximum age of the universe as a function of mass-energy density [4].

**Table 1:** Maximum possible age of the universe as a function of mass-energy density

Matter density ( $10^{-31} \text{ gm cm}^{-3}$ )	$t_{\text{max}}, 10^{10} \text{ years}$
1	9.43
4	4.72
8	3.34
12	2.72
16	2.36
20	2.11
30	1.72
40	1.49
60	1.22
80	1.05
100	0.94

The total time elapsed from the big bang singularity in FRW model at  $t = 0$  to the present epoch is introduced by a spacelike hyper surface  $S_0$  which gives the age of the universe [3,11];

$$t_{\text{age}} = H_0^{-1} f(q_0) \quad (39)$$

where  $H_0$  and  $q_0$  are the Hubble constant and deceleration parameters respectively. Here  $f(q_0)$  is a monotonic decreasing function of  $q_0$  is given as follows:

$$f(q_0) = (1-2q_0)^{-1} - q_0(1-2q_0)^{-3/2} \cosh^{-1}\left(\frac{1}{q_0} - 1\right),$$

$$0 < q_0 < \frac{1}{2},$$

$$f(q_0) = \frac{2}{3}, \quad q_0 = \frac{1}{2},$$

$$f(q_0) = \frac{q_0}{(2q_0 - 1)^{3/2}} \left[ \cos^{-1}\left(\frac{1}{q_0} - 1\right) - \frac{(2q_0 - 1)^{3/2}}{q_0} \right]$$

$$\frac{1}{2} < q_0 < \infty.$$

Here  $f(q_0) \rightarrow 1$  at  $q_0 \rightarrow 0$  and  $f(q_0) \rightarrow 0$  at  $q_0 \rightarrow \infty$ .

The observational data are extremely uncertain regarding the value of  $q_0$  and  $H_0$ , so  $t_{\text{age}}$  is that of  $H_0$  lies between  $40-120 \text{ km}^{-1} \text{ Mpc}^{-1}$ . This corresponds to the upper bounds on age is given by;

$$t_{\text{max}} = \begin{cases} 24.4 \times 10^9 \text{ year for } H_0 = 40 \text{ km s}^{-1} \text{ Mpc}^{-1} \\ 8.1 \times 10^9 \text{ year for } H_0 = 120 \text{ km s}^{-1} \text{ Mpc}^{-1} \end{cases} \quad (41)$$

Comparing (41) with Table 1 we see that universe need not be exactly Friedmann type, and,  $H_0$  and  $q_0$  are not required which are anyway uncertain at the moment.

## 6. CONCLUSIONS

In this paper we have tried to describe the general upper limit of the age of the universe. We have described related topics of general relativity thinking for the common readers. The universe is homogeneous and isotropic around us about 14 billion light years. In our discussion we have not strictly followed the homogeneity and isotropy of the universe to determine the age of the universe. We have applied various types of models to find the present age of the universe. In every case we have found the figure is around  $10^{10}$  years. We have avoided difficult mathematical calculations and have displayed diagrams where necessary.

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