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A New Class of Bianchi Type-I Cosmological Models with Viscosity and Cosmological Term

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***Abstract*—Exact solutions of Einstein's field equations for a Bianchi type-I space-time filled with a dissipative fluid and cosmological term Λ are investigated. To get deterministic solution we choose the scale factor $a(t) = \sqrt{t^n e^t}$, where n is a positive constant. This choice of the scale factor yields a time dependent deceleration parameter (DP), representing a model which generates a transition of the universe from the early decelerating phase to the recent accelerating phase. In this paper we generalize the recent results of Yadav et al. [27]. We find a variety of solutions in Bianchi type-I space-time for different values of n with variable and constant viscosity and cosmological constant. The cosmological constant $\Lambda(t)$ is found to be in good agreement with recent supernovae Ia observations. The cosmic jerk parameter is also found to be in good concordance with the recent data of astrophysical observations under appropriate condition. The physical and geometric properties of the derived models are discussed in the light of thermodynamic.**

***Index Terms*—Bianchi type-I models; Entropy; Variable deceleration parameter; Cosmological constant**
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I. Introduction

In general relativity the cosmological constant Λ may be regarded as the measure of energy density of the vacuum and can, in principle, lead to the avoidance of the big-bang singularity, which is a characteristic of other Friedmann-Robertson-Walker (FRW) models. The cosmological constant problem has a long history, and while there are many possible resolutions, none has gained widespread acceptance. In classical general relativity, the energy density and pressure of the vacuum obeys the relation $\rho c^2 = p = \Lambda c^4/8\pi G$, where c is the speed of light and G is the gravitational constant. The astrophysical-determined value of Λ, for the present epoch at least, is small but in quantum theory, the vacuum (or zero-point) energies associated with particle interactions lead to a value of Λ which is big. This discrepancy may be large as 10^{120}. Padmanabhan [1], [2] have recently reviewed this problem, and outlined a resolution wherein the classical value of Λ is essentially the statistical value âleft overâ from numerous stronger interactions described by quantum field theory.

In Einsteinâs field equations, the non-trivial role of vacuum generates a cosmological constant Λ-term which leads to the inflationary scenario [3] which predicted that during an early exponential phase, the vacuum energy is treated as large cosmological constant which is expected by Glashow-Salam-Weinberg and by Grand Unified Theory as mentioned by Langacker [4]. Therefore, the present day observations of smallness of cosmological constant $\Lambda \leq 10^{-56} cm^{-2}$ support to assume that cosmological constant is time dependent. The observation of the Supernovae of type Ia [5], [6] provides the evidence that the universe is undergoing accelerated expansion. Relating the observations from Cosmic Background Radiation [7], [8] and SDSS [9], [10], one concludes that the universe at present is dominated by 70% exotic component, dubbed dark energy, which has negative pressure and push the universe to accelerated expansion. Of course, a natural explanation to the accelerated expansion is due to a positive tiny cosmological constant. Though, it suffers the so-called fine tuning and cosmic coincidence problems. However, in 2σ confidence level, it fits the observations very well [11]. If the cosmological constant is not a real constant but is time variable, the fine tuning and cosmic coincidence problems can be removed. In fact, this possibility was considered in the past years. This problem is already discussed by several authors [12−16] in detail. Bertolami [17] investigated that $\Lambda \propto t^{-2}$ plays a significant role in cosmological studies. Chen and Wu [18] have suggested that cosmological constant $\Lambda \propto R^{-2}$ plays a significant role in the cosmological and astrophysical studies.

Dissipative effects including both the bulk and shear viscosity, play a significant role in the early evolution of the universe. The first attempts to create a theory of relativistic dissipative fluids, were made by Esckart [19] and Landau & Lifshitz [20]. Weinberg [21] derived general formula for bulk and shear viscosity, and used them to evaluate the cosmological entropy production rate. Grøn [22] reviewed viscous cosmological models and studied the role of viscosity in evolution of the universe in Bianchi type-I space-time. Recently, Pradhan and Lata [23], Pradhan et al. [24]–[26] and Yadav et al. [27] have studied the Bianchi types cosmological models with viscosity in different contexts.

Recently, Yadav et al. [27] investigated Bianchi type-I cosmological models with viscosity and cosmological term in general relativity by considering a scale factor $a(t) = \sqrt{te^t}$ to solve the field equations. In this paper, we generalize the previous solutions [27] by considering the scale factor $a(t) = \sqrt{t^n e^t}$, which yields a time dependent deceleration parameter (DP), representing a model which generates a transition of the universe from the early decelerating phase to the recent accelerating phase. In this paper, we have investigated a new class of spatially homogeneous and anisotropic Bianchi type-I cosmological models with time dependent deceleration parameter and cosmological constant in presence of a dissipative fluid. The Einstein's field equations are solved explicitly. The outline of the paper is as follows: In Sect. 2, the basic equations are described. Section 3 deals with the solutions of the field equations by considering time dependent deceleration parameter. Section 4 describes results and discussions. In Subsect. 4.1, we obtain the solution with variable Λ-term and constant ξ and also discuss the thermodynamic equations. Subsection 4.2 deals with the models with variable Λ-term and $\xi \propto \rho$ and also discuss entropy production rate of model. In Subsect. 4.3, we describe the solution with constant Λ-term and time dependent ξ along with the thermodynamic equation and its aspects of the models. Section 5 deals with cosmic jerk parameter. Finally, conclusions are summarized in the last Sect. 6.

II. The Basic equations

We consider a spatially homogeneous and anisotropic Bianchi type-I metric in the form

$$ds^2 = -dt^2 + A^2(t)dx^2 + B^2(t)dy^2 + C^2(t)dz^2, \quad (1)$$

where the metric potentials A, B and C are functions of cosmic time t alone. This ensures that the model is spatially homogeneous.

We define the following parameters to be used in solving Einstein's field equations for t he metric (1).

The average scale factor a of Bianchi type-I model (1) is defined as

$$a = (ABC)^{\frac{1}{3}}. \quad (2)$$

A volume scale factor V is given by

$$V = a^3 = ABC. \quad (3)$$

In analogy with FRW universe, we also define the generalized Hubble parameter H and deceleration parameter q as

$$H = \frac{\dot{a}}{a} = \frac{1}{3}\left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right), \quad (4)$$

$$q = -\frac{a\ddot{a}}{\dot{a}^2} = -\frac{\ddot{a}}{aH^2}, \quad (5)$$

where an over dot denotes derivative with respect to the cosmic time t.

Also we have

$$H = \frac{1}{3}(H_1 + H_2 + H_3), \quad (6)$$

where $H_1 = \frac{\dot{A}}{A}$, $H_2 = \frac{\dot{B}}{B}$ and $H_3 = \frac{\dot{C}}{C}$ are directional Hubble factors in the directions of x-, y- and z-axes respectively.

The Einstein's field equations (in gravitational unit $8\pi G = c = 1$) are given by

$$R_{ij} - \frac{1}{2}Rg_{ij} = -T_{ij}, \quad (7)$$

where T_{ij} is the stress energy tensor of matter which, in case of viscous fluid and cosmological constant, has the form [20]

$$T_{ij} = (\rho + \bar{p})u_i u_j + \bar{p}g_{ij} - \eta\mu_{ij} - \Lambda(t)g_{ij}, \quad (8)$$

with

$$\bar{p} = p - \left(\xi - \frac{2}{3}\eta\right)u^i_{;i} = p - (3\xi - 2\eta)H \quad (9)$$

and

$$\mu_{ij} = u_{i;j} + u_{j;i} + u_i u^\alpha u_{j;\alpha} + u_j u^\alpha u_{i;\alpha}. \quad (10)$$

In the above equations, ξ and η stand for the bulk and shear viscosity coefficients respectively; ρ is the matter density; p is the isotropic pressure and u^i is the four-velocity vector satisfying $u^i u_i = -1$.

In a co-moving coordinate system, where $u^i = \delta^i_0$, the field equations (7), for the anisotropic Bianchi type-I space-time (1) and viscous fluid distribution (8), yield

$$\frac{\ddot{B}}{B} + \frac{\ddot{C}}{C} + \frac{\dot{B}\dot{C}}{BC} = -\bar{p} + 2\eta\frac{\dot{A}}{A} + \Lambda, \quad (11)$$

$$\frac{\ddot{C}}{C} + \frac{\ddot{A}}{A} + \frac{\dot{C}\dot{A}}{CA} = -\bar{p} + 2\eta\frac{\dot{B}}{B} + \Lambda, \quad (12)$$

$$\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\dot{A}\dot{B}}{AB} = -\bar{p} + 2\eta\frac{\dot{C}}{C} + \Lambda, \quad (13)$$

$$\frac{\dot{A}\dot{B}}{AB} + \frac{\dot{B}\dot{C}}{BC} + \frac{\dot{C}\dot{A}}{CA} = \rho + \Lambda. \quad (14)$$

Here, and also in what follows, a dot designates ordinary differentiation with respect to t.

Equations (11)-(14) can also be written as

$$p - \xi\theta - \Lambda = H^2(2q - 1) - \sigma^2, \quad (15)$$

$$\rho + \Lambda = 3H^2 - \sigma^2, \quad (16)$$

where σ is shear scalar given by

$$\sigma^2 = \frac{1}{2}\sigma_{ij}\sigma^{ij} = \frac{1}{2}\left[\left(\frac{\dot{A}}{A}\right)^2 + \left(\frac{\dot{B}}{B}\right)^2 + \left(\frac{\dot{C}}{C}\right)^2\right] - \frac{1}{6}\theta^2, \quad (17)$$

where

$$\sigma_{ij} = u_{i;j} + \frac{1}{2}(u_{i;k}u^k u_j + u_{j;k}u^k u_i) + \frac{1}{3}\theta(g_{ij} + u_i u_j). \quad (18)$$

The expansion scalar (θ) and the anisotropy parameter (A_m) are defined as

$$\theta = u^i_{;i} = \frac{3\dot{a}}{a} \quad (19)$$

$$A_m = \frac{1}{3}\sum_{i=1}^{3}\left(\frac{H_i - H}{H}\right)^2. \quad (20)$$

The energy conservation equation $T^{ij}_{\ \ ;j} = 0$, leads to the following expression:

$$\dot{\rho} = -(p + \rho)\theta + \xi\theta^2 + 4\eta\sigma^2 - \dot{\Lambda}. \quad (21)$$

It follows from (21) that for contraction, that is, $\theta < 0$, we have $\dot{\rho} > 0$ so that the matter density increases or decreases depending on whether the viscous heating is greater or less than the cooling due to expansion.

The Raychaudhuri equation is obtained as

$$\dot{\theta} = -\frac{1}{2}[\rho + 3(p - \xi\theta)] - \frac{1}{3}\theta^2 - 2\sigma^2 + \Lambda. \tag{22}$$

We have a system of four independent equations (11)−(14) and eight unknown variables, namely A, B, C, p, ρ, ξ, η and Λ. So for complete determinacy of the system, we need four appropriate relations among these variables that we shall consider in the following section and solve the field equations.

III. Solutions of field equations

Subtracting (11) from (12), (11) from (13), (12) from (13) and taking second integral of each, we get the following three relations

$$\frac{A}{B} = d_1 \exp\left(x_1 \int a^{-3} e^{-2\int \eta dt} dt\right), \tag{23}$$

$$\frac{A}{C} = d_2 \exp\left(x_2 \int a^{-3} e^{-2\int \eta dt} dt\right), \tag{24}$$

$$\frac{B}{C} = d_3 \exp\left(x_3 \int a^{-3} e^{-2\int \eta dt} dt\right), \tag{25}$$

where d_1, x_1, d_2, x_2, d_3 and x_3 are constants of integration.

From (23)−(25), the metric functions can be explicitly written as

$$A(t) = a_1 \; a \exp\left(b_1 \int a^{-3} e^{-2\int \eta dt} dt\right), \tag{26}$$

$$B(t) = a_2 \; a \exp\left(b_2 \int a^{-3} e^{-2\int \eta dt} dt\right), \tag{27}$$

$$C(t) = a_3 \; a \exp\left(b_3 \int a^{-3} e^{-2\int \eta dt} dt\right), \tag{28}$$

where

$$a_1 = \sqrt[3]{d_1 d_2}, \quad a_2 = \sqrt[3]{d_1^{-1} d_3}, \quad a_3 = \sqrt[3]{(d_2 d_3)^{-1}},$$

$$b_1 = \tfrac{x_1 + x_2}{3}, \quad b_2 = \tfrac{x_3 - x_1}{3}, \quad b_3 = \tfrac{-(x_2 + x_3)}{3}.$$

These constants satisfy the following two relations

$$a_1 a_2 a_3 = 1, b_1 + b_2 + b_3 = 0. \tag{29}$$

Thus the metric functions are found explicitly in terms of the average scale factor a. It is clear from Eqs. (26)−(28) that once we get the value of the average scale factor a, we can easily calculate the metric functions A, B, C.

Following Pradhan and Amirhashchi [28], Saha et al. [29] and Pradhan et al. [30], [31], we consider the DP as a variable for which we consider the variation of scale factor a with cosmic time t by the relation

$$a(t) = \sqrt{t^n e^t}, \tag{30}$$

n is an arbitrary constant. This *ansatz* generalized the one proposed by Amirhashchi et al. [32], Pradhan et al . [33] and Yadav et al. [27]. Recently, Pradhan [34] examined the relation (30) when studying accelerating dark energy models with anisotropic fluid in Bianchi type-VI_0 space-time. If we put $n = 0$ in Eq. (30), it is reduced to $a(t) = \sqrt{e^t}$, i.e. an exponential law of variation for the scale factor. Thus, our choice of scale factor is physically acceptable. It is worth mentioning here that one can also select many other ansatzes than Eq. (30) which mimic an accelerating universe. However, one should also be careful to check the physical acceptability and stability of their corresponding solutions, otherwise they do not prove any relation of such solutions with the observable universe. Equation (30) yields physically plausible solutions.

From (5) and (30), we get the time varying DP as

$$q(t) = \frac{2n}{(n+t)^2} - 1. \tag{31}$$

From Eq. (31), we observe that $q > 0$ for $t < \sqrt{2n} - n$ and $q < 0$ for $t > \sqrt{2n} - n$. It is observed that for $0 < n < 2$, our model is evolving from deceleration phase to acceleration phase. Also, recent observations of SNe Ia, expose that the present universe is accelerating and the value of DP lies to some place in the range $-1 < q < 0$. It follows that in our derived model, one can choose the value of DP consistent with the observation. Figure 1 graphs the deceleration parameter (q) versus time which gives the behaviour of q from decelerating to accelerating phase for different values of n. Thus our derived models have accelerated expansion at present epoch which is consistent with recent observations of Type Ia supernova [5], [6] and CMB anisotropies [35]–[37].

Next, we assume that the coefficient of shear viscosity (η) is proportional to the expansion scalar (θ) i.e. $\eta \propto \theta$, which leads to

$$\eta = \eta_0 \theta, \tag{32}$$

where η_0 is proportionality constant. Such relation has already been proposed in the physical literature as a physically plausible relation [38]–[40].

Finally to conveniently specify the source, we assume the perfect gas equation of state, which may be written as

$$p = \gamma\rho, 0 \leq \gamma \leq 1. \tag{33}$$

Using Eqs. (30) and (32) into (26)−(28), we get the following expressions for the scale factors

$$A = a_1 \sqrt{t^n e^t} \exp\left[b_1 \int \left[\left(t^n e^t\right)^{-\frac{3}{2}(1+2\eta_0)}\right] dt\right], \tag{34}$$

$$B = a_2 \sqrt{t^n e^t} \exp\left[b_2 \int \left[\left(t^n e^t\right)^{-\frac{3}{2}(1+2\eta_0)}\right] dt\right], \tag{35}$$

$$C = a_3 \sqrt{t^n e^t} \exp\left[b_3 \int \left[\left(t^n e^t\right)^{-\frac{3}{2}(1+2\eta_0)}\right] dt\right]. \tag{36}$$

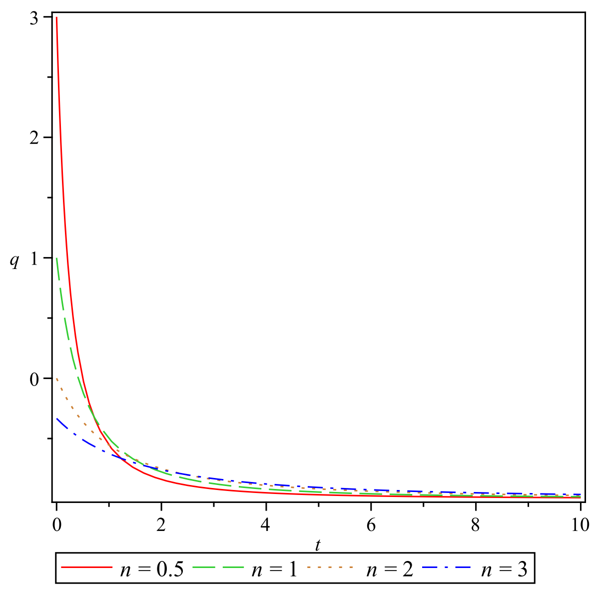

Figure 1. The plot of deceleration parameter q versus t

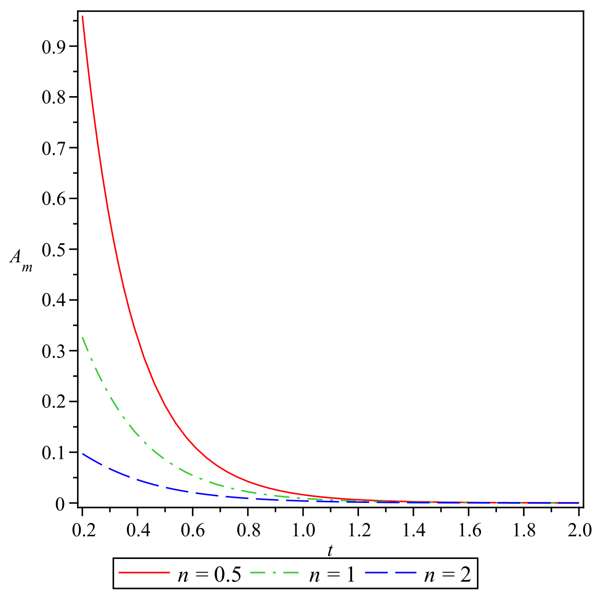

Figure 2. The plot of anisotropy parameter A_m versus t. Here $\eta_0 = 0.1$, $\beta_1 = 1$

IV. Results and discussions

The physical parameters such as directional Hubble factors (H_i), Hubble parameter (H), expansion scalar (θ), spatial volume (V), deceleration parameter (q), anisotropy parameter (A_m) and shear scalar (σ) are given by

$$H_i = \frac{1}{2}\left(\frac{n}{t}+1\right) + b_i \left(t^n e^t\right)^{-\frac{3}{2}(2\eta_0+1)}, \tag{37}$$

$$H = \frac{1}{2}\left(\frac{n}{t}+1\right), \tag{38}$$

$$\theta = \frac{3}{2}\left(\frac{n}{t}+1\right), \tag{39}$$

$$V = \left(t^n e^t\right)^{\frac{3}{2}}, \tag{40}$$

$$A_m = \frac{4}{3}\beta_1 \left(\frac{t}{n+t}\right)^2 \left(t^n e^t\right)^{-3(2\eta_0+1)}, \tag{41}$$

$$\sigma^2 = \frac{1}{2}\beta_1 \left(t^n e^t\right)^{-3(2\eta_0+1)}, \tag{42}$$

where

$$\beta_1 = b_1^2 + b_2^2 + b_3^2,$$
$$\beta_2 = b_1 b_2 + b_2 b_3 + b_3 b_1,$$
$$\beta_3 = b_2^2 + b_3^2 + b_2 b_3. \tag{43}$$

The shear viscosity of the model reads as

$$\eta = \frac{3}{2}\eta_0 \left(\frac{n}{t}+1\right). \tag{44}$$

Equations (15) and (16) lead to

$$p - \frac{3}{2}\xi\left(\frac{n}{t}+1\right) - \Lambda = \frac{1}{4}\left(\frac{n}{t}+1\right)^2 \left\{\frac{4n}{(n+2)^2} - 3\right\} - \frac{1}{2}\beta_1 \left(t^n e^t\right)^{-3(2\eta_0+1)}, \tag{45}$$

$$\rho + \Lambda = \frac{3}{4}\left(\frac{n}{t}+1\right)^2 - \frac{1}{2}\beta_1 \left(t^n e^t\right)^{-3(2\eta_0+1)}, \tag{46}$$

From Eqs. (40) and (39), we observe that the spatial volume is zero at $t = 0$ and the expansion scalar is infinite, which show that the universe starts evolving with zero volume at $t = 0$ which is a big bang scenario. From Eqs. (34)–(36), we observe that the spatial scale factors are zero at the initial epoch $t = 0$ and hence the model has a point type singularity [41]. We observe that proper volume increases exponentially as time increases. Thus, the models represent the inflationary scenario.

The dynamics of the mean anisotropic parameter depends on

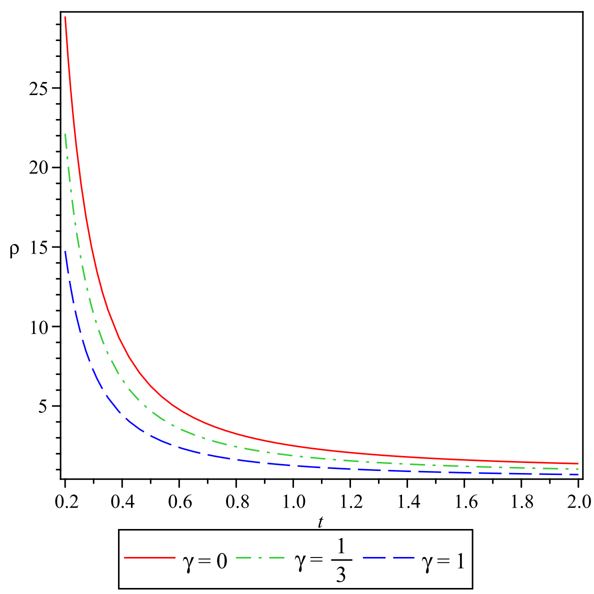

Figure 3. The plot of energy density ρ versus t for $\xi_0 = \eta_0 = 0.1$, $\beta_2 = \beta_3 = 0.5$, $n = 1$

the constant $\beta_1 = b_1^2 + b_2^2 + b_3^2$. From Eq. (41), we observe that at late time when $t \to \infty$, $A_m \to 0$. Thus, our model

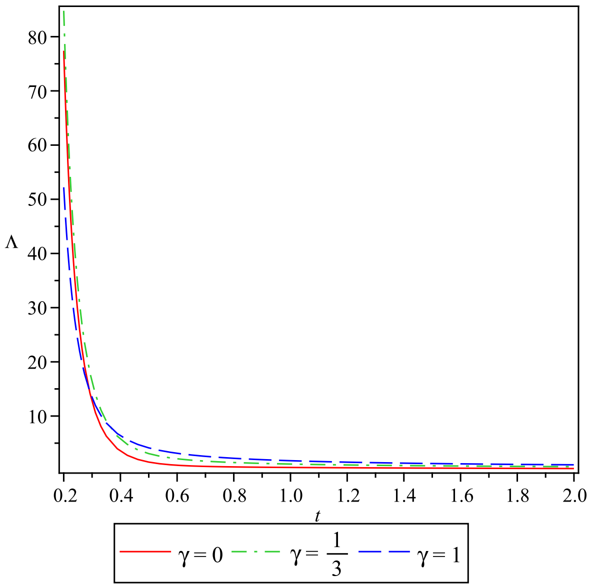

Figure 4. The plot of cosmological constant Λ versus t for $\xi_0 = \eta_0 = 0.1$, $\beta_2 = \beta_3 = 0.5$, $n = 1$

has transition from initial anisotropy to isotropy at present epoch which is in good harmony with current observations. Figure 2 depicts the variation of anisotropy parameter (A_m) versus cosmic time t. From the figure, we observe that A_m decreases with time and tends to zero as $t \to \infty$. Thus, the observed isotropy of the universe can be achieved in our model at present epoch.

Here, we solve the Eqs. (45) and (46) with (33) in the following cases:

A. Models with variable Λ-term and constant ξ

Let us assume that the coefficient of bulk viscosity is constant, i.e. $\xi(t) = \xi_0 =$ constant. Then the Eqs. (45) and (46) together with (33) leads the following expressions for energy density, pressure and cosmological constant:

$$\rho = \frac{1}{(1+\gamma)}\left[(\beta_2 - \beta_3)\left(t^n e^t\right)^{-3(2\eta_0+1)} + \frac{n}{t^2} + \frac{3}{2}\xi_0\left(\frac{n}{t}+1\right)\right], \quad (47)$$

$$p = \frac{\gamma}{(1+\gamma)}\left[(\beta_2 - \beta_3)\left(t^n e^t\right)^{-3(2\eta_0+1)} + \frac{n}{t^2} + \frac{3}{2}\xi_0\left(\frac{n}{t}+1\right)\right], \quad (48)$$

$$\Lambda = \frac{3}{4}\left(\frac{n}{t}+1\right)^2 + \frac{1}{(1+\gamma)}\left[(\beta_2\gamma + \beta_3)\left(t^n e^t\right)^{-3(2\eta_0+1)} - \frac{n}{t^2} - \frac{3}{2}\xi_0\left(\frac{n}{t}+1\right)\right]. \quad (49)$$

From above relations (47)−(49), we can obtain four types of models:

- When $\gamma = 0$, we obtain empty model.
- When $\gamma = \frac{1}{3}$, we obtain radiating dominated model.
- When $\gamma = -1$, we have the degenerate vacuum or false vacuum or ρ vacuum model [42].
- When $\gamma = 1$, the fluid distribution corresponds with the equation of state $\rho = p$ which is known as Zeldovich fluid or stiff fluid model [43], [44].

From Eq. (47), it is observed that the energy density ρ is a decreasing function of time and $\rho > 0$ always. The energy density has been graphed versus time in Figure 3 for $\gamma = 0, \frac{1}{3}, 1$. It is apparent that the energy density remains positive in all three types of models. However, it decreases more sharply with the cosmic time in Zeldovich universe, compare to radiating dominated and empty fluid universes. Also it can be seen from the figure that ρ decreases more sharply with time in radiating dominated universe, compare to empty universe.

Figure 4 describes the variation of cosmological term Λ with time (ρ and Λ are in geometric units in entire paper) for $\gamma = 0, \frac{1}{3}, 1$. This is taken to be a representative case of physical viability of the models. In all three types of models, we observe that Λ is decreasing function of time t and it approaches a small positive value at late time (i.e. at present epoch). However, it decreases more sharply with the cosmic time in empty universe, compare to radiating dominated and stiff fluid universes. The Λ-term also decreases more sharply in radiating dominated universe, compare to stiff fluid universe. A positive cosmological constant resists the attractive gravity of matter due to its negative pressure and drives the accelerated expansion of the universe. Recent cosmological observations [5]− [10] suggest the existence of a positive cosmological constant Λ with the magnitude $\Lambda(G\hbar/c^3) \approx 10^{-123}$. These observations on magnitude and red-shift of type Ia supernova suggest that our universe may be an accelerating one with induced cosmological density through the cosmological Λ-term. Thus, the nature of Λ in our derived models are supported by recent observations.

1) Thermodynamic equations: The energy in a comoving volume is $U = \rho V$. The equation for production of entropy S in a comoving volume due to the dissipative effects in a fluid with temperature $\mathbf{T}$ is given by

$$\mathbf{T}\dot{S} = \dot{U} + p\dot{V} = 3V(3\xi + 2\eta A_m)H^2. \quad (50)$$

In a cosmic fluid where the energy density and pressure of the cosmic fluid are functions of temperature only, $\rho = \rho(\mathbf{T})$, $p = p(\mathbf{T})$ and where the cosmic fluid has no net charge, we obtain easily [22]

$$S = \frac{V}{\mathbf{T}}(\rho + p). \quad (51)$$

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From (50) and (51), we get the following expression for the entropy production rate in viscous Bianchi type-I universe

$$\frac{\dot{S}}{S} = \frac{3(3\xi + 2\eta A_m)H^2}{\rho + p}. \tag{52}$$

For a fluid obeying the equation of state (33), Eqs. (51) and (52) reduce to

$$S = \frac{V}{\mathbf{T}}(1+\gamma)\rho, \tag{53}$$

$$\frac{\dot{S}}{S} = \frac{3(3\xi + 2\eta A_m)H^2}{(1+\gamma)\rho}. \tag{54}$$

Equation (54) can be rewritten as

$$\frac{\dot{S}}{S} = \frac{\xi + 4\eta(\sigma^2/\theta^2)}{(1+\gamma)(\rho/\theta^2)}, \tag{55}$$

which gives the rate of change of entropy with time.

Let the entropy density be s so that

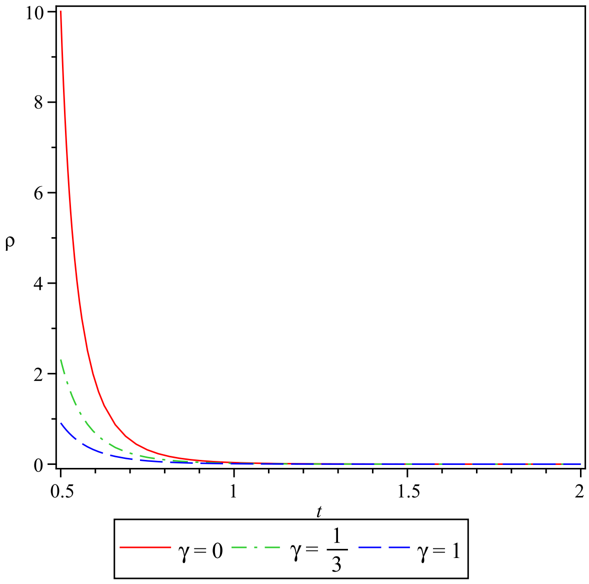

Figure 5. The plot of energy density ρ versus t for $\xi_0 = \eta_0 = 0.1$, $n = 1$, $\beta_4 = 0.5$

$$s = \frac{S}{V} = \frac{(1+\gamma)\rho}{\mathbf{T}}. \tag{56}$$

It defines the entropy density in terms of the temperature.

The first law of thermodynamics may be written as

$$d(\rho V) + \gamma\rho dV = (1+\gamma)\mathbf{T}d\left(\frac{\rho V}{\mathbf{T}}\right), \tag{57}$$

which on integration, yields

$$\mathbf{T} \sim \rho^{\left(\frac{\gamma}{1+\gamma}\right)}. \tag{58}$$

From (56) and (57), one can get

$$s \sim \rho^{\left(\frac{1}{1+\gamma}\right)}. \tag{59}$$

The entropy in a comoving volume then varies according to

$$S \sim sV. \tag{60}$$

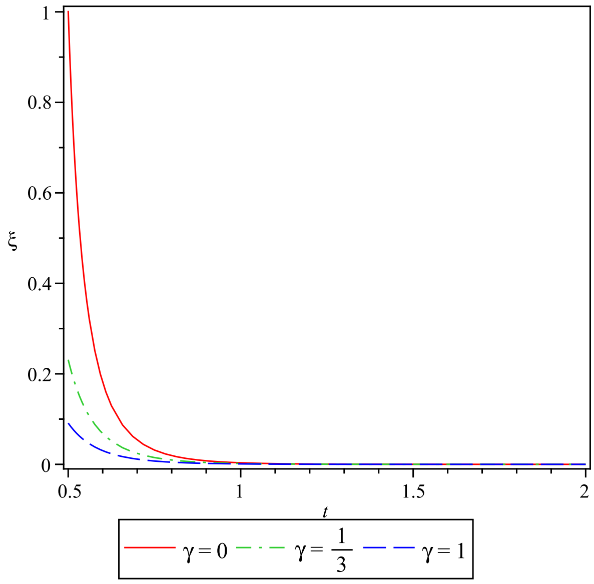

Figure 6. The plot of bulk viscosity coefficient ξ versus t for $\xi_0 = \eta_0 = 0.1$, $n = 1$, $\beta_4 = 0.5$

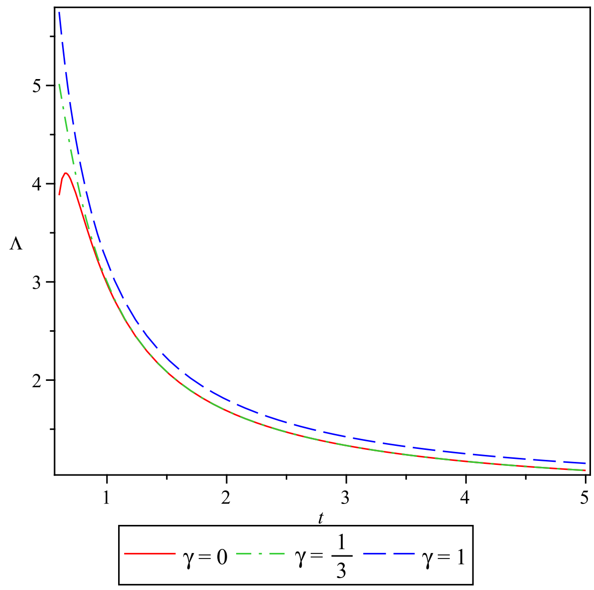

Figure 7. The plot of cosmological constant Λ versus t for $\xi_0 = \eta_0 = 0.1$, $n = 1$, $\beta_2 = \beta_4 = 0.5$

These equations are not valid for a vacuum fluid with $\gamma = -1$. For a Zel'dovich fluid ($\gamma = 1$), we get

$$\mathbf{T} \sim \rho^{\frac{1}{2}} \quad \text{and} \quad s \sim \rho^{\frac{1}{2}}, \tag{61}$$

so that the entropy density is proportional to the temperature.

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Using (40) and (47), we find the respective temperature ($\mathbf{T}$), entropy density (s) and total entropy (S) from (58)−(60), as

$$\mathbf{T} = \mathbf{T}_0\left[\frac{1}{(1+\gamma)}\left\{(\beta_2-\beta_3)\left(t^n e^t\right)^{-3(2\eta_0+1)} + \frac{n}{t^2} + \frac{3}{2}\xi_0\left(\frac{n}{t}+1\right)\right\}\right]^{\frac{\gamma}{1+\gamma}}, \tag{62}$$

$$s = s_0\left[\frac{1}{(1+\gamma)}\left\{(\beta_2-\beta_3)\left(t^n e^t\right)^{-3(2\eta_0+1)} + \frac{n}{t^2} + \frac{3}{2}\xi_0\left(\frac{n}{t}+1\right)\right\}\right]^{\frac{1}{1+\gamma}}, \tag{63}$$

$$S = S_0\left(t^n e^t\right)^{\frac{3}{2}}\left[\frac{1}{(1+\gamma)}\left\{(\beta_2-\beta_3)\left(t^n e^t\right)^{-3(2\eta_0+1)} + \frac{n}{t^2} + \frac{3}{2}\xi_0\left(\frac{n}{t}+1\right)\right\}\right]^{\frac{1}{1+\gamma}}, \tag{64}$$

where $\mathbf{T}_0$, S_0 and s_0 are positive constants.

Now the entropy production rate of the model is given by

$$\frac{\dot{S}}{S} = \frac{3\left[\frac{3}{4}\xi_0\left(\frac{n}{t}+1\right)^2 + \eta_0\beta_1\left(\frac{n}{t}+1\right)\left(t^n e^t\right)^{-3(2\eta_0+1)}\right]}{(\beta_2-\beta_3)\left(t^n e^t\right)^{-3(2\eta_0+1)} + \frac{n}{t^2} + \frac{3}{2}\xi_0\left(\frac{n}{t}+1\right)}. \tag{65}$$

From above equation (65), we observe that the rate of change of entropy with time, i.e. the relation $\frac{\dot{S}}{S} > 0$. This also implies that the total entropy increases with time in Bianchi type-I model presented in this paper. This is in good agreement with second law of thermodynamics.

B. Models with variable Λ-term and $\xi \propto \rho$

Let us consider that $\xi = \xi_0\rho$. In this case we obtain the expressions for energy density, pressure, bulk viscosity and cosmological constant as follows:

$$\rho = \frac{1}{\left\{1+\gamma-3\left(\frac{n}{t}+1\right)\right\}} \times \left[\beta_4\left(t^n e^t\right)^{-3(2\eta_0+1)}\right], \tag{66}$$

$$p = \frac{\gamma}{\left\{1+\gamma-3\left(\frac{n}{t}+1\right)\right\}} \times \left[\beta_4\left(t^n e^t\right)^{-3(2\eta_0+1)}\right], \tag{67}$$

$$\xi = \frac{\xi_0}{\left\{1+\gamma-3\left(\frac{n}{t}+1\right)\right\}} \times \left[\beta_4\left(t^n e^t\right)^{-3(2\eta_0+1)}\right], \tag{68}$$

$$\Lambda = \frac{3}{4}\left(\frac{n}{t}+1\right)^2 + \left[\beta_2 - \frac{\beta_4}{\left\{1+\gamma-3\xi_0\left(\frac{n}{t}+1\right)\right\}}\right]\left(t^n e^t\right)^{-3(2\eta_0+1)}, \tag{69}$$

where

$$\beta_4 = b_1b_2 + b_3b_1 - b_2^2 - b_3^2.$$

In this case, we find the respective temperature ($\mathbf{T}$), entropy density (s) and total entropy (S) as given by

$$\mathbf{T} = \frac{\mathbf{T}_0}{\left\{1+\gamma-3\xi_0\left(\frac{n}{t}+1\right)\right\}} \times \left[\beta_4\left(t^n e^t\right)^{-3(2\eta_0+1)}\right]^{\frac{\gamma}{1+\gamma}} \tag{70}$$

$$s = \frac{s_0}{\left\{1+\gamma-3\xi_0\left(\frac{n}{t}+1\right)\right\}} \times \left[\beta_4\left(t^n e^t\right)^{-3(2\eta_0+1)}\right]^{\frac{1}{1+\gamma}} \tag{71}$$

$$S = \frac{S_0\left(t^n e^t\right)^{\frac{3}{2}}}{\left(1+\gamma-3\xi_0\left(\frac{n}{t}+1\right)\right)} \times \left[\beta_4\left(t^n e^t\right)^{-3(2\eta_0+1)}\right]^{\frac{1}{1+\gamma}} \tag{72}$$

The expression for entropy production rate of model is given by

$$\frac{\dot{S}}{S}=\frac{3\left[\frac{3}{4}\xi_0\left(\frac{n}{t}+1\right)^2+\eta_0\beta_1\left(\frac{n}{t}+1\right)\left(t^ne^t\right)^{-3(2\eta_0+1)}\right]}{\frac{(1+\gamma)}{\left\{1+\gamma-\frac{3}{2}\left(\frac{n}{t}+1\right)\right\}}\left[\beta_4\left(t^ne^t\right)^{-3(2\eta_0+1)}\right]}. \tag{73}$$

Figures 5 and 7 depict the variation of energy density ρ and cosmological term Λ versus time for $\gamma=0,\frac{1}{3},1$ respectively. The nature of ρ and Λ in this model are same as described in previous section 3.1.

Figure 6 plots the variation of bulk viscosity coefficient ξ with time t. From this figure we observe that ξ is a positive decreasing function of time and it approaches to a constant quantity which is near to zero for all three types of models $\gamma=0,\frac{1}{3},1$. This is in good agreement with physical behaviour of ξ. However, it decreases more sharply with the cosmic time in empty universe, compare to radiating dominated and Zeldovich universes. It is also observed from the figure that ξ decreases more sharply with time in radiating dominated universe, compare to empty universe.

From Eq. (73), we observe that $\frac{\dot{S}}{S}>0$. This implies that the total entropy increases with time. This is reproducible with second law of thermodynamics.

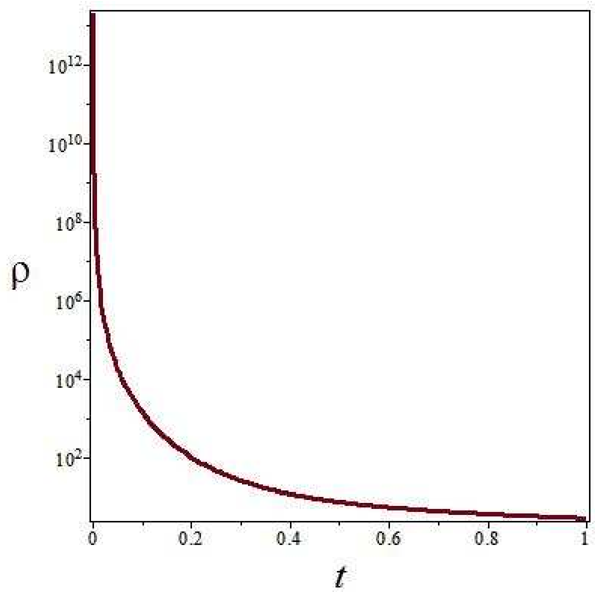

Figure 8. The plot of energy density ρ versus t for $\eta_0=0.1$, $n=1$, $\beta_2=0.5$, $\Lambda=0$

C. Models with constant Λ-term and $\xi(T)$

In this case, the expressions for energy density, isotropic pressure and bulk viscosity coefficient are respectively, given by

$$\rho=\frac{3}{4}\left(\frac{n}{t}+1\right)^2+\beta_2\left(t^ne^t\right)^{-3(2\eta_0+1)}-\Lambda, \tag{74}$$

$$p=\gamma\left[\frac{3}{4}\left(\frac{n}{t}+1\right)^2+\beta_2\left(t^ne^t\right)^{-3(2\eta_0+1)}-\Lambda\right], \tag{75}$$

$$\xi=\frac{(1+\gamma)(n+t)}{2t}+\frac{2\beta_4\,t}{3(n+t)}\left(t^ne^t\right)^{-3(2\eta_0+1)}-\frac{2n}{3t(n+t)}-\frac{2(1+\gamma)\Lambda\,t}{3(n+t)}. \tag{76}$$

In this case, we find the respective temperature ($\mathbf{T}$), entropy density (s), total entropy (S) and entropy production rate of model are given by

$$\mathbf{T}=\mathbf{T}_0\left[\frac{3}{4}\left(\frac{n}{t}+1\right)^2+\beta_2\left(t^ne^t\right)^{-3(2\eta_0+1)}-\Lambda\right]^{\frac{\gamma}{(1+\gamma}}, \tag{77}$$

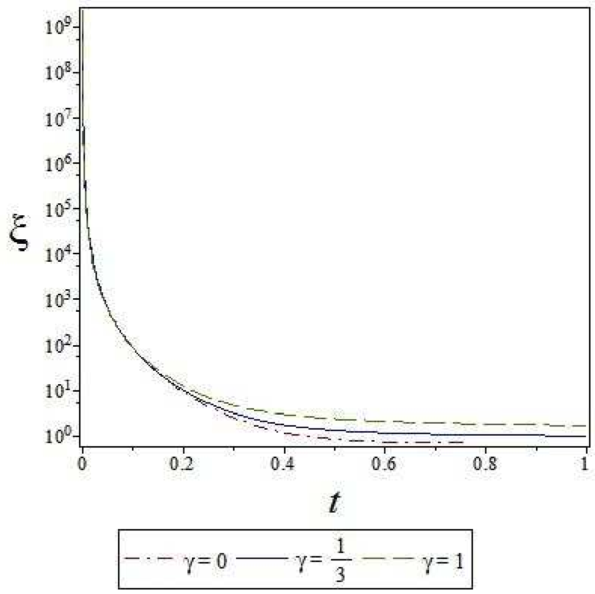

Figure 9. The plot of bulk viscosity coefficient ξ versus t for $\eta_0 = 0.1$, $n = 1$, $\beta_4 = 0.5$

$$s = s_0 \left[\frac{3}{4}\left(\frac{n}{t}+1\right)^2 + \beta_2 \left(t^n e^t\right)^{-3(2\eta_0+1)} - \Lambda\right]^{\frac{1}{(1+\gamma}}, \tag{78}$$

$$S = S_0 \left(t^n e^t\right)^{\frac{3}{2}} \left[\frac{3}{4}\left(\frac{n}{t}+1\right)^2 + \beta_2 \left(t^n e^t\right)^{-3(2\eta_0+1)} - \Lambda\right]^{\frac{1}{(1+\gamma}}, \tag{79}$$

$$\frac{\dot{S}}{S} = \frac{\frac{3}{4}\left(\frac{n+t}{t}\right)\left[\frac{3}{2}(1+\gamma)\left(\frac{n+t}{t}\right)^2 + 2(\beta_4 + 2\eta_0\beta_1)\left(t^n e^t\right)^{-3(2\eta_0+1)} - \frac{2n}{t^2} - 2(1+\gamma)\Lambda\right]}{(1+\gamma)\left[\frac{3}{4}\left(\frac{n}{t}+1\right)^2 + \beta_2 \left(t^n e^t\right)^{-3(2\eta_0+1)} - \Lambda\right]} \tag{80}$$

From Eq. (74), it is observed that the energy density ρ is a decreasing function of time and $\rho > 0$ always. Figure 8 depicts the variation of energy density with cosmic time which shows this nature of energy density.

Figure 9 plots the variation of bulk viscosity coefficient ξ with time t. From this figure we observe that ξ is a positive decreasing function of time and it approaches to a constant quantity which is near to zero for all three types of models $\gamma = 0, \frac{1}{3}, 1$. This is in good agreement with physical behaviour of ξ. However, it decreases more sharply with the cosmic time in empty universe, compare to radiating dominated and Zeldovich universes. It is also observed from the figure that ξ decreases more sharply with time in radiating dominated universe, compare to empty universe.

From Eq. (80), we observe that $\frac{\dot{S}}{S} > 0$. This implies that the total entropy increases with time. This is reproducible with second law of thermodynamics.

V. Cosmic Jerk Parameter

A convenient method to describe models close to Λ CDM is based on the cosmic jerk parameter j, a dimensionless third derivative of the scale factor with respect to the cosmic time [45]− [49]. A deceleration-to-acceleration transition occurs for models with a positive value of j_0 and negative q_0. Flat Λ CDM models have a constant jerk $j = 1$. The jerk parameter in cosmology is defined as the dimensionless third derivative of the scale factor with respect to cosmic time

$$j(t) = \frac{1}{H^3}\frac{\dddot{a}}{a}. \tag{81}$$

and in terms of the scale factor to cosmic time

$$j(t) = \frac{\left(a^2H^2\right)''}{2H^2}. \tag{82}$$

where the 'dots' and 'primes' denote derivatives with respect to cosmic time and scale factor, respectively. One can rewrite Eq. (81) as

$$j(t) = q + 2q^2 - \frac{\dot{q}}{H}. \tag{83}$$

Eqs. (31) and (83) reduce to

$$j(t) = 1 - \frac{6n}{(n+t)^2} + \frac{8n}{(n+1)^3}. \tag{84}$$

This value overlaps with the value $j \simeq 2.16$ obtained from the combination of three kinematical data sets: the gold sample of type Ia supernovae [50], the SNIa data from the SNLS project [51], and the X-ray galaxy cluster distance measurements [52] for

$$t = 3.45 \times 10^{-2}\kappa - \frac{50n}{\kappa} - n, \tag{85}$$

where $\kappa = 10^4 n[8.41 + 1.45\sqrt{(14.4n + 33.6)}]^{\frac{1}{3}}$. If we put $n = 1$ in above expressions, we obtain the results of Jerk Parameter obtained by Yadav et al. [27].

VI. Discussion and Summary

In this paper, a class of cosmological models are presented with variable deceleration parameter q and cosmological term Λ in spatially homogeneous and anisotropic Bianchi type-I space-time in presence of bulk and shear viscosity. To find the explicit solution, we have considered a time dependent deceleration parameter which yields a scale factor as $a(t) = \sqrt{t^n e^t}$, n being a positive constant. One of the most intriguing aspects of this evolution is the recently established late-time transition from a decelerated to an accelerating regime of the expansion of the Universe. This shows that in the early stage the universe was decelerating where as the universe is accelerating at present epoch which is corroborated from the recent supernovae Ia observation [5−10]. The parameter H_i, H, θ, and σ diverge at the initial singularity. There is a Point Type singularity [41] at $t = 0$ in the model. The rate of expansion slows down and finally tends to zero as $t \to 0$. The pressure, energy density and scalar field become negligible whereas the scale factors and spatial volume become infinitely large as $t \to \infty$, which would give essentially an empty universe.

The main features of the models are as follows:

- The models are based on exact solutions of the Einstein's field equations for the anisotropic Bianchi-I space-time in presence of a dissipative fluid with variable Λ-term.
- The model represents expanding, accelerating, shearing and non-rotating universe.
- Our solutions generalize the results recently obtained by Yadav et al. [27]. If we put $n = 1$ in our derived models, we obtain all models obtained in [27].

- For different choice of n, we can generate a class of cosmological models in Bianchi type-I space-time. It is observed that such models are also in good harmony with current observations. Thus, the solutions demonstrated in this paper may be useful for better understanding of the characteristic of anisotropic models with viscosity and cosmological Λ-term in the evolution of the universe within the framework of Bianchi type-I space-time.
- The cosmological constant has been assumed to represent the energy density of vacuum, which has a potentially important contribution in the dynamics of the evolution of universe. The cosmological constant is observed to have a small and positive value at late times. The nature of decaying vacuum energy density $\Lambda(t)$ in our derived models are supported by recent cosmological observations. These observations on magnitude and red-shift of type Ia supernova [5−10] suggest that our universe may be an accelerating one with induced cosmological density through the cosmological Λ-term.
- The present models have a transition of the universe from the early deceleration phase to the recent acceleration phase (see, Figure 1) which is in good agreement with recent observations [53].
- The shear viscosity is observed to be responsible for the faster removal of initial anisotropies in the universe. This can be seen from the expression of anisotropy parameter (41). Hence, the isotropy observed in the present universe, is a possible consequence of viscous effects in the cosmic fluid.

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