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On Certain Subclass of p-valent Meromorphically Starlikes Functions with Alternating Coefficients

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ABSTRACT

A certain subclass $B_m(p,\alpha,\lambda,\ell,A,B)$ consisting of meromorphic p-valent functions with alternating coefficient in $U^*=\{z\ :\ z\in C\ :\ 0<|z|<1\}$ is introduced. In this paper we obtain coefficient inequalities, distortion theorem, closure theorems and class preserving integral operators for functions in the class $B_m(p,\alpha,\lambda,\ell,A,B)$ are obtained.

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1. INTRODUCTION

Let $\Sigma(p)$ denote the class of functions of the form:

$$f(z)=\frac{a_{-p}}{z^p}+\sum_{k=1}^{\infty}a_kz^k\quad (a_{-p}\neq 0;\ p\in N:=\{1,3,5,.....\}),\tag{1.1}$$

which are regular in the punctured disc

$$U^*=\{z\ :\ z\in C\ :\ 0<|z|<1\}=U\setminus\{0\}.$$

Definition 1:
Let f,g be analytic in U. Then g is said to be subordinate to f, written $g\prec f$, if there exists a Schwarz function $w(z)$, which is analytic in U, with $w(0)=0$ and $|w(z)|<1\,(z\in U)$ such that $g(z)=f(w(z))\quad (z\in U)$. Hence $g(z)\prec f(z)\,(z\in U)$, then $g(0)=f(0)$ and $g(U)\subset f(U)$. In particular, if the function $f(z)$ is univalent in U, we have the following (e.g.[1];[2]): $g(z)\prec f(z)\quad (z\in U)$ if and only if $g(0)=f(0)$ and $g(U)\subset f(U)$.

Definition 2:
For functions $f(z)\in\sum(p)$ given by (1.1) and $g(z)\in\sum(p)$ defined by

$$g(z)=\frac{b_{-p}}{z^p}+\sum_{k=1}^{\infty}b_kz^k\quad (b_k\geq 0,p\in N),\tag{1.2}$$

we define the convolution (or Hadamard product) of $f(z)$ and $g(z)$ by

$$(f*g)(z)=\frac{a_{-p}b_{-p}}{z^p}+\sum_{k=1}^{\infty}a_kb_kz^k\quad (p\in N,z\in U).\tag{1.3}$$

Now, using the integral operator
$L_p^m(\lambda,\ell)\quad (\ell>0;\lambda\geq 0;\ p\in N;m\in N_0;z\in U^*)$
Introduced by El-Ashwah [3], the for function $f(z)\in\sum(p)$ given by (1.1) as follows:

$$L_p^m(\lambda,\ell)f(z)=\frac{a_{-p}}{z^p}+\sum_{k=1}^{\infty}\left[\frac{\ell}{\ell+\lambda(k+p)}\right]^m a_kz^k.\tag{1.4}$$

It is easily verified from (1.4) that

$$\lambda z(L_p^{m+1}(\lambda,\ell))f(z))'=\ell L_p^m(\lambda,\ell)f(z)-(\ell+p\lambda)L_p^{m+1}(\lambda,\ell)f(z)\ (\lambda>0),\tag{1.5}$$

We note that:

i. $L_p^{\alpha}(1,1)f(z)=p_p^{\alpha}f(z)$ (see Aqlan et al. [4]);
ii. $L_1^{\alpha}(1,\beta)f(z)=p_{\beta}^{\alpha}f(z)(a_{-p}=1)$ (see Lashin [5]);
iii. $L(1,\gamma)f(z)=J(f(z))(p=1)$ (see Sh. Najafzadeh [6]);
iv. $L_p^m(1,\alpha)f(z)=J_{p,\alpha}^m f(z)$ (see El -Ashwah et. at [7]).

Let $B_m(p,\alpha,\lambda,\ell,A,B)$ denote the class of functions $f(z)$ in $\sum(p)$ that satisfy the condition:

$$\frac{\ell L_p^m(\lambda,\ell)f(z)}{\lambda L_p^{m+1}(\lambda,\ell)f(z)}-\left(\frac{\lambda p+\ell}{\lambda}\right)\prec-\frac{p+[pB+(A-B)(p-\alpha)]z}{1+Bz};z\in U^*\tag{1.6}$$

Where $\prec$ denotes subordination, $0\leq\alpha<p$, $-1\leq A<B\leq 1$, $0<B\leq 1$, $\lambda,\ell>0$, $p\in N$ and $m\in N_0$.
By definition of subordination, the condition (1.6) is equivalent to

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$$\frac{\ell L_p^m(\lambda,\ell)f(z)}{\lambda L_p^{m+1}(\lambda,\ell)f(z)}-\left(\frac{\lambda p+\ell}{\lambda}\right)=\frac{p+[pB+(A-B)(p-\alpha)]w(z)}{1+Bw(z)}, \quad \textbf{(1.7)}$$

Where $w(z)\in H=\{w$ regular, $w(0)=0$ and $|w(z)|<1, z\in U\}$. it is easy to see that the condition (1.7) is equivalent to

$$\left|\frac{\dfrac{\ell L_p^m(\lambda,\ell)f(z)}{\lambda L_p^{m+1}(\lambda,\ell)f(z)}-\dfrac{\ell}{\lambda}}{B\left[\dfrac{\ell L_p^m(\lambda,\ell)f(z)}{\lambda L_p^{m+1}(\lambda,\ell)f(z)}-\left(\dfrac{\lambda p+\ell}{\lambda}\right)\right]+[pB+(A-B)(p-\alpha)]}\right|<1 \ (z\in U^*). \quad \textbf{(1.8)}$$

We note that:

i. When $A=-1, B=1$, we have $f(z)\in B_m(p,\alpha,\lambda,\ell)$ if

$$\operatorname{Re}\left\{\frac{\ell L_p^m(\lambda,\ell)f(z)}{\lambda L_p^{m+1}(\lambda,\ell)f(z)}-\left(\frac{\lambda p+\ell}{\lambda}\right)\right\}<-\alpha,$$

ii. when $\alpha=0$

$$\frac{\ell L_p^m(\lambda,\ell)f(z)}{\lambda L_p^{m+1}(\lambda,\ell)f(z)}-\left(\frac{\lambda p+\ell}{\lambda}\right)\prec-\frac{1+Az}{1+Bz}.$$

Let $\sum_a(p)$ be the subclass of $\sum(p)$ consists of functions of the form:

$$f(z)=\frac{a_{-p}}{z^p}+\sum_{k=1}^{\infty}(-1)^{k-1}a_k z^k\,(a_{-p}\neq 0; a_k\geq 0; p\in N), \quad \textbf{(1.9)}$$

that are regular and p-valent in U^*

Let us write

$$\sum_a^*(p,n,\alpha,\lambda,\ell,A,B)=B_m(p,\alpha,\lambda,\ell,A,B)\cap\sum_a(p). \quad \textbf{(1.10)}$$

In this paper coefficient inequalities, distortion theorem and closure theorems for the class $B_m^*(p,\alpha,\lambda,\ell,A,B)$ are obtained. Finally the class preserving integral operators of the form

$$F(z)=(c-p+1)z^{-c-1}\int_0^z t^c f(t)dt \quad (c>p-1; p\in N), \quad \textbf{(1.11)}$$

is considered. Techniques used are similar to those of Silverman [8] and Uralegaddi and Ganigi [9], Aouf and Darwish [10], Aouf and Hossen and Lashin [11].

2. COEFFICIENT INEQUALITIES

Theorem 1:

Let $f(z)=\dfrac{a_{-p}}{z^p}+\sum_{k=1}^{\infty}a_k z^k$ be regular and p-valent in U^*. If

$$\sum_{k=1}^{\infty}\left[\frac{\ell}{\ell+\lambda(k+p)}\right]^{m+1}[(k+p)(1+B)+(A-B)(p-\alpha)]|a_k|\leq(B-A)(p-\alpha)|a_{-p}|, \quad \textbf{(2.1)}$$

then $f(z)\in B_m(p,\alpha,\lambda,\ell,A,B)$.

Proof:

Suppose (2.1) holds for all admissible values of $p,m,\alpha,\lambda,\ell,A$ and B. It suffices show that

$$\left|\frac{\dfrac{L_p^m(\lambda,\ell)f(z)}{L_p^{m+1}(\lambda,\ell)f(z)}-1}{B\left[\dfrac{L_p^m(\lambda,\ell)f(z)}{L_p^{m+1}(\lambda,\ell)f(z)}-(\frac{\lambda}{\ell}p+1)\right]+\lambda/\ell[pB+(A-B)(p-\alpha)]}\right|<1 \quad \text{for} \quad |z|<1$$

We have

$$\left|\frac{\dfrac{L_p^m(\lambda,\ell)f(z)}{L_p^{m+1}(\lambda,\ell)f(z)}-1}{B\left[\dfrac{L_p^m(\lambda,\ell)f(z)}{L_p^{m+1}(\lambda,\ell)f(z)}-(\frac{\lambda}{\ell}p+1)\right]+\lambda/\ell[pB+(A-B)(p-\alpha)]}\right|$$

$$= \left| \frac{\sum_{k=1}^{\infty} \left[\frac{\ell}{\ell + \lambda(k+p)} \right]^{m+1} (k+p) a_k z^{p+k}}{(A-B)(p-\alpha) a_{-p} + \sum_{k=1}^{\infty} \left[\frac{\ell}{\ell + \lambda(k+p)} \right]^{m+1} [B(k+p) + (A-B)(p-a)] a_k z^{k+p}} \right|$$

$$\leq \frac{\sum_{k=1}^{\infty} \left[\frac{\ell}{\ell + \lambda(k+p)} \right]^{m+1} (k+p) |a_k|}{(B-A)(p-\alpha)|a_{-p}| - \sum_{k=1}^{\infty} \left[\frac{\ell}{\ell + \lambda(k+p)} \right]^{m+1} [B(k+p) + (A-B)(p-a)] |a_k|}.$$

The last expression is bounded above by (1.1) if

$$\sum_{k=1}^{\infty} \left[\frac{\ell}{\ell + \lambda(k+p)} \right]^{m+1} (k+p)|a_k| \leq (B-A)(p-a)|a_{-p}| -$$

$$\sum_{k=1}^{\infty} \left[\frac{\ell}{\ell + \lambda(k+p)} \right]^{m+1} [B(k+p) + (A-B)(p-\alpha)] |a_k|$$

which is equivalent to

$$\sum_{k=1}^{\infty} \left[\frac{\ell}{\ell + \lambda(k+p)} \right]^{m+1} [(k+p)(1+B) + (A-B)(p-\alpha)] |a_k| \leq (B-A)(p-\alpha)|a_{-p}|.$$

The completes the proof of Theorem 1.

Theorem 2: Let

$$f(z) = \frac{a_{-p}}{z^p} + \sum_{k=1}^{\infty} (-1)^{k-1} a_k z^k \, (a_{-p} \neq 0; a_k \geq 0, p \in N)$$

be regular and p-valent in U^*. Then $f(z) \in \sum_a^* (p, m, \alpha, \lambda, \ell, A, B)$ if and only if

$$\sum_{k=1}^{\infty} \left[\frac{\ell}{\ell + \lambda(k+p)} \right]^{m+1} [(k+p)(1+B) + (A-B)(p-\alpha)] a_k$$

$$\leq (B-A)(p-\alpha)|a_{-p}|. \qquad \textbf{(2.2)}$$

Proof:

In view of Theorem 1, it is sufficient to show the " only if " part. Let us suppose that

$$f(z) = \frac{a_{-p}}{z^p} + \sum_{k=1}^{\infty} (-1)^{k-1} a_k z^k \, (a_{-p} \neq 0; a_k \geq 0, p \in N)$$

is in $\sum_a^* (p, m, \alpha, \lambda, \ell, A, B)$. Then

$$\left| \frac{\frac{L_p^m(\lambda, \ell) f(z)}{L_p^{m+1}(\lambda, \ell) f(z)} - 1}{B \left[\frac{L_p^{m+1}(\lambda, \ell) f(z)}{L_p^{m+1}(\lambda, \ell) f(z)} - \left(\frac{\lambda}{\ell} p + 1 \right) \right] + \frac{\lambda}{\ell} [pB + (A-B)(p-\alpha)]} \right|$$

$$= \left| \frac{\sum_{k=1}^{\infty} (-1)^{k-1} \left[\frac{\ell}{\ell + \lambda(k+p)} \right]^{m+1} (k+p) a_k z^{k+p}}{(B-A)(p-\alpha) a_{-p} - \sum_{k=1}^{\infty} (-1)^{k-1} \left[\frac{\ell}{\ell + \lambda(k+p)} \right]^{m+1} [B(k+p) + (A-B)(p-\alpha)] a_k z^{k+p}} \right| \leq 1$$

for all $z \in U^*$. Using the fact that $\text{Re}(z) \leq |z|$ for all z, it follows that

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$$\operatorname{Re}\left\{\frac{\sum_{k=1}^{\infty}(-1)^{k-1}\left[\frac{\ell}{\ell+\lambda(k+p)}\right]^{m+1}(k+p)a_k z^{k+p}}{(B-A)(p-\alpha)a_{-p}-\sum_{k=1}^{\infty}(-1)^{k-1}\left[\frac{\ell}{\ell+\lambda(k+p)}\right]^{m+1}\left[B(k+p)+(A-B)(p-\alpha)\right]a_k z^{k+p}}\right\}\le 1 \qquad (z\in U^*). \tag{2.3}$$

Now choose values of z on the real axis so that $\frac{L_p^m(\lambda,\ell)f(z)}{L_p^{m+1}(\lambda,\ell)f(z)}$ is real. Upon clearing the denominator in (2.3) and letting $z\to 1^-$ through real values, we obtain

$$\sum_{k=1}^{\infty}\left[\frac{\ell}{\ell+\lambda(k+p)}\right]^{m+1}(k+p)a_k \le (B-A)(p-\alpha)\left|a_{-p}\right| -$$

$$\sum_{k=1}^{\infty}\left[\frac{\ell}{\ell+\lambda(k+p)}\right]^{m+1}[B(k+p)+(A-B)(p-\alpha)]a_k .$$

or

$$\left[\frac{\ell}{\ell+\lambda(k+p)}\right]^{m+1}\left[(k+p)(1+B)+(A-B)(p-\alpha)\right]a_k$$

$$\le (B-A)(p-\alpha)\left|a_{-p}\right|.$$

Hence the result follows .

Corollary 1:

If $f(z)=\frac{a_{-p}}{z^p}+\sum_{k=1}^{\infty}(-1)^{k-1}a_k z^k$ $(a_{-p}\ne 0; a_k\ge 0; p\in N)$ is in the class $\sum_a^*(p,\alpha,\lambda,\ell,A,B)$, then

$$a_k \le \frac{(B-A)(p-\alpha)\left|a_{-p}\right|}{\left[\frac{\ell}{\ell+\lambda(k+p)}\right]^{m+1}\left[(k+p)(1+B)+(A-B)(p-\alpha)\right]} \qquad (k\in N). \tag{2.4}$$

Equality holds for the functions of form

$$f_k(z)=\frac{a_{-p}}{z^p}+(-1)^{k-1}\frac{(B-A)(p-\alpha)a_{-p}}{\left[\frac{\ell}{\ell+\lambda(k+p)}\right]^{m+1}\left[(k+p)(1+B)+(A-B)(p-\alpha)\right]}z^k \quad (k\in N;\, p\in N). \tag{2.5}$$

3. DISTORTION THEOREMS

Theorem 3:

Let $f(z)=\frac{a_{-p}}{z^p}+\sum_{k=1}^{\infty}(-1)^{k-1}a_k z^k$ $(a_{-p}\ne 0; a_k\ge 0; p\in N)$ is in the class $\sum_a^*(p,m,\alpha,\lambda,\ell,A,B)$, then for $0<|z|=r<1$,

$$\frac{\left|a_{-p}\right|}{r^p}-\frac{(B-A)(p-\alpha)\left|a_{-p}\right|}{\left[\frac{\ell}{\ell+\lambda(1+p)}\right]^{m+1}\left[(1+p)(1+B)+(A-B)(p-\alpha)\right]}r \le \left|f(z)\right|$$

$$\le \frac{\left|a_{-p}\right|}{r^p}+\frac{(B-A)(p-\alpha)\left|a_{-p}\right|}{\left[\frac{\ell}{\ell+\lambda(1+p)}\right]^{m+1}\left[(1+p)(1+B)+(A-B)(p-\alpha)\right]}r \tag{3.1}$$

with equality for the function

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$$f_1(z)=\frac{a_{-p}}{z^p}+\frac{(B-A)(p-\alpha)|a_{-p}|}{\left[\frac{\ell}{\ell+\lambda(1+p)}\right]^{m+1}\left[(1+p)(1+B)+(A-B)(p-\alpha)\right]}z \quad \text{at} \quad z=r,ir\,. \qquad \textbf{(3.2)}$$

Proof:

Suppose that $f(z)$ in the class $\sum_a^*(p,m,\alpha,\lambda,\ell,A,B)$.

In view of Theorem 2, we have

$$\left[\frac{\ell}{\ell+\lambda(1+p)}\right]^{m+1}\left[(1+p)(1+B)+(A-B)(p-\alpha)\right]\sum_{k=1}^{\infty}a_k$$

$$\le\sum_{k=1}^{\infty}\left[\frac{\ell}{\ell+\lambda(k+p)}\right]^{m+1}\left[(1+p)(1+B)+(A-B)(p-\alpha)\right]a_k$$

$$\le (B-A)(p-\alpha)|a_{-p}|,$$

which evidently yields

$$\sum_{k=1}^{\infty}a_k\le\frac{(B-A)(p-\alpha)|a_{-p}|}{\left[\frac{\ell}{\ell+\lambda(1+p)}\right]^{m+1}\left[(1+p)(1+B)+(A-B)(p-\alpha)\right]}. \qquad \textbf{(3.3)}$$

Consequently, we obtain

$$|f(z)|\le\frac{|a_{-p}|}{r^p}+\sum_{k=1}^{\infty}a_k r^k\le\frac{|a_{-p}|}{r^p}+r\sum_{k=1}^{\infty}a_k$$

$$\le\frac{|a_{-p}|}{r^p}+\frac{(B-A)(p-\alpha)|a_{-p}|}{\left[\frac{\ell}{\ell+\lambda(1+p)}\right]^{m+1}\left[(1+p)(1+B)+(A-B)(p-\alpha)\right]}r,$$

by (3.3). This gives the right hand inequality of (3.1). Also

$$|f(z)|\ge\frac{|a_{-p}|}{r^p}-\sum_{k=1}^{\infty}a_k r^k\ge\frac{|a_{-p}|}{r^p}-r\sum_{k=1}^{\infty}a_k\ge\frac{|a_{-p}|}{r^p}-\frac{(B-A)(p-\alpha)|a_{-p}|}{\left[\frac{\ell}{\ell+\lambda(1+p)}\right]^{m+1}\left[(1+p)(1+B)+(A-B)(p-\alpha)\right]}r,$$

which gives the left hand side of (3.1). It can be easily seen that the function $f(z)$ defined by (3.2) is an extremely function for the theorem.

Theorem 4: If $f(z)=\frac{a_{-p}}{z^p}+\sum_{k=1}^{\infty}(-1)^{k-1}a_k z^k$ $(a_{-p}\ne 0;a_k\ge 0,\,p\in N)$ is in the class $\sum_a^*(p,m,\alpha,\lambda,\ell,A,B)$, then for $0<|z|=r<1$,

$$\frac{p|a_{-p}|}{r^{p+1}}-\frac{(B-A)(p-\alpha)|a_{-p}|}{\left[\frac{\ell}{\ell+\lambda(1+p)}\right]^{m+1}\left[(1+p)(1+B)+(A-B)(p-\alpha)\right]}\le|f'(z)|\le$$

$$\frac{p|a_{-p}|}{r^{p+1}}+\frac{(B-A)(p-\alpha)|a_{-p}|}{\left[\frac{\ell}{\ell+\lambda(1+p)}\right]^{m+1}\left[(1+p)(1+B)+(A-B)(p-\alpha)\right]}. \qquad \textbf{(3.4)}$$

The result is sharp, the extremely function being of the form (3.2).

Proof:
From Theorem 2, we have

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$$\left[\frac{\ell}{\ell+\lambda(1+p)}\right]^{m+1}\left[(1+p)(1+B)+(A-B)(p-\alpha)\right]\sum_{k=1}^{\infty} ka_k$$

$$\leq \sum_{k=1}^{\infty}\left[\frac{\ell}{\ell+\lambda(1+p)}\right]^{m+1}\left[(1+p)(1+B)+(A-B)(p-\alpha)\right]a_k \leq (B-A)(p-\alpha)\left|a_{-p}\right|,$$

Which evidently yields

$$\sum_{k=1}^{\infty} ka_k \leq \frac{(B-A)(p-\alpha)\left|a_{-p}\right|}{\left[\frac{\ell}{\ell+\lambda(1+p)}\right]^{m+1}\left[(1+p)(1+B)+(A-B)(p-\alpha)\right]} \tag{3.5}$$

Consequently, we obtain

$$\left|f'(z)\right| \leq \frac{p\left|a_{-p}\right|}{r^{p+1}} + \sum_{k=1}^{\infty} ka_k r^{k-1} \leq \frac{p\left|a_{-p}\right|}{r^{p+1}} + \sum_{k=1}^{\infty} ka_k \leq \frac{p\left|a_{-p}\right|}{r^{p+1}}$$

$$+ \frac{(B-A)(p-\alpha)\left|a_{-p}\right|}{\left[\frac{\ell}{\ell+\lambda(1+p)}\right]^{m+1}\left[(1+p)(1+B)+(A-B)(p-\alpha)\right]},$$

by (3.5). Also

$$\left|f'(z)\right| \geq \frac{p\left|a_{-p}\right|}{r^{p+1}} - \sum_{k=1}^{\infty} ka_k r^{k-1} \geq \frac{p\left|a_{-p}\right|}{r^{p+1}} - \sum_{k=1}^{\infty} ka_k$$

$$\geq \frac{p\left|a_{-p}\right|}{r^{p+1}} - \frac{(B-A)(p-\alpha)\left|a_{-p}\right|}{\left[\frac{\ell}{\ell+\lambda(1+p)}\right]^{m+1}\left[(1+p)(1+B)+(A-B)(p-\alpha)\right]}. \tag{3.6}$$

This completes the proof of Theorem 4.

4. CLOSURE THEOREMS

Let the functions $f_j(z)$ be defined for $j = 1, 2, ..., m$, by

$$f_j(z) = \frac{a_{-p,j}}{z^p} + \sum_{k=1}^{\infty}(-1)^{k-1}a_{k,j}z^k \quad (a_{-p,j} > 0; a_{k,j} \geq 0; p \in N) \quad \text{for } z \in U^*. \tag{4.1}$$

Theorem 5:
Let the function $f_j(z)$ be defined by (4.1) be in the class $\sum_a^*(p,\alpha,\lambda,\ell,A,B)$ for every $j = 1, 2, ..., m$. Then the function $F(z)$ defined by

$$F(z) = \frac{b_{-p}}{z^p} + \sum_{k=1}^{\infty}(-1)^{k-1}b_k z^k (b_{-p} > 0; b_k \geq 0; p \in N), \tag{4.2}$$

is a member of the class $\sum_a^*(p,\alpha,\lambda,\ell,A,B)$, where

$$b_{-p} = \frac{1}{m}\sum_{j=1}^{m} a_{-p,j} \text{ and } b_k = \frac{1}{m}\sum_{j=1}^{m} a_{k,j} \; (k \in N). \tag{4.3}$$

Proof:

Since $f_j(z) \in \sum_a^*(p,\alpha,\lambda,\ell,A,B)$, it follows from Theorem 2 that

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$$\sum_{k=1}^{\infty}\left[\frac{\ell}{\ell+\lambda(k+p)}\right]^{m+1}\left[(k+p)(1+B)+(A-B)(p-\alpha)\right]a_{k,j}\le(B-A)(p-\alpha)\left|a_{-p,j}\right| \tag{4.4}$$

for every $j=1,2,....,m$. Hence,

$$\sum_{k=1}^{\infty}\left[\frac{\ell}{\ell+\lambda(k+p)}\right]^{m+1}\left[(k+p)(1+B)+(A-B)(p-\alpha)\right]b_k$$

$$=\sum_{k=1}^{\infty}\left[\frac{\ell}{\ell+\lambda(k+p)}\right]^{m+1}\left[(k+p)(1+B)+(A-B)(p-\alpha)\right]\left\{\frac{1}{m}\sum_{j=1}^{\infty}a_{k,j}\right\}$$

$$=\frac{1}{m}\sum_{j=1}^{m}\left(\sum_{k=1}^{\infty}\left[\frac{\ell}{\ell+\lambda(k+p)}\right]^{m+1}\left[(k+p)(1+B)+(A-B)(p-\alpha)\right]a_{k,j}\right)$$

$$\le(B-A)(p-\alpha)\left(\frac{1}{m}\sum_{j=1}^{\infty}a_{-p,j}\right)=(B-A)(p-\alpha)b_{-p},$$

which (in view of Theorem 2) implies that $F(z)\in\sum_a^*(p,\alpha,\lambda,\ell,A,B).$

Theorem 6:

The class $\sum_a^*(p,\alpha,\lambda,\ell,A,B)$ is closed under convex linear combination.

Proof:

Let the functions $f_j(z)\quad(j=1,2)$ defined by (4.1) be in the class $\sum_a^*(p,\alpha,\lambda,\ell,A,B),$ it is sufficient to prove that the function

$$H(z)=tf_1(z)+(1-t)f_2(z)\quad(0\le t\le 1) \tag{4.5}$$

is also in the class $\sum_a^*(p,\alpha,\lambda,\ell,A,B).$ Since for $0\le t\le 1,$

$$H(z)=\frac{ta_{-p,1}+(1-t)a_{-p,2}}{z^p}+\sum_{k=1}^{\infty}(-1)^{k-1}\{ta_{k,1}+(1-t)a_{k,2}\}z^k, \tag{4.6}$$

we observe that

$$\sum_{k=1}^{\infty}\left[\frac{\ell}{\ell+\lambda(k+p)}\right]^{m+1}\left[(k+p)(1+B)+(A-B)(p-\alpha)\right]\left\{ta_{k,1}+(1-t)a_{k,2}\right\}$$

$$=t\sum_{k=1}^{\infty}\left[\frac{\ell}{\ell+\lambda(k+p)}\right]^{m+1}\left[(k+p)(1+B)+(A-B)(p-\alpha)\right]a_{k,1}\ +$$

$$(1-t)\sum_{k=1}^{\infty}\left[\frac{\ell}{\ell+\lambda(k+p)}\right]^{m+1}\left[(k+p)(1+B)+(A-B)(p-\alpha)\right]a_{k,2}$$

$$\le(B-A)(p-\alpha)\left\{ta_{-p,1}+(1-t)a_{-p,2}\right\}, \tag{4.7}$$

which the aid of Theorem 2, hence $H(z)\in\sum_a^*(p,\alpha,\lambda,\ell,A,B)$. This completes the proof of Theorem 5.

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Theorem 7: Let $f_0(z)=\frac{a_{-p}}{z^p}$ **(4.8)**

and

$$f_k(z)=\frac{a_{-p}}{z^p}+(-1)^{k-1}\frac{(B-A)(p-\alpha)|a_{-p}|}{\left[\frac{\ell}{\ell+\lambda(k+p)}\right]^{m+1}\{(k+p)(1+B)+(A-B)(p-\alpha)\}}z^k\,(k\in N). \quad \textbf{(4.9)}$$

Then $f(z)\in\sum_a^*(p,\alpha,\lambda,\ell,A,B)$ if and only if it can be expressed in the form

$$f(z)=\sum_{k=0}^{\infty}\mu_k f_k(z),\ \text{where}\,\mu_k\geq 0\ \ (k\geq 0)\ \text{ and }\ \sum_{k=0}^{\infty}\mu_k=1. \quad \textbf{(4.10)}$$

Proof: Suppose that

$$f(z)=\sum_{k=0}^{\infty}\mu_k f_k(z),\ \text{where}\ \mu_k\geq 0\ \ (k\geq 0)\ \text{ and }\sum_{k=0}^{\infty}\mu_k=1.\ \text{Then}$$

$$f(z)=\sum_{k=0}^{\infty}\mu_k f_k(z)$$

$$=\mu_0 f_0(z)+\sum_{k=1}^{\infty}\mu_k f_k(z)$$

$$=\frac{a_{-p}}{z^p}+\sum_{k=1}^{\infty}\mu_k(-1)^{k-1}\frac{(B-A)(p-\alpha)|a_{-p}|}{\left[\frac{\ell}{\ell+\lambda(k+p)}\right]^{m+1}[(k+p)(1+B)+(A-B)(p-\alpha)]}z^k\,(k\in N). \quad \textbf{(4.11)}$$

Since

$$\sum_{k=1}^{\infty}\left[\frac{\ell}{\ell+\lambda(k+p)}\right]^{m+1}\{(k+p)(1+B)+(A-B)(p-\alpha)\}.$$

$$\frac{(B-A)(p-\alpha)|a_{-p}|}{\left[\frac{\ell}{\ell+\lambda(k+p)}\right]^{m+1}\{(k+p)(1+B)+(A-B)(p-\alpha)\}}\mu_k$$

$$=(B-A)(p-\alpha)|a_{-p}|\sum_{k=1}^{\infty}\mu_k$$

$$=(B-A)(p-\alpha)|a_{-p}|(1-\mu_0)$$

$$\leq(B-A)(p-\alpha)|a_{-p}|\quad, \quad \textbf{(4.12)}$$

we have $f(z)\in\sum_a^*(p,\alpha,\lambda,\ell,A,B)$, by Theorem 2 .

Conversely, suppose that the function $f(z)$ defined by (1.9) belongs to the class $\sum_a^*(p,\alpha,\lambda,\ell,A,B)$. Since

$$a_k\leq\frac{(B-A)(p-\alpha)|a_{-p}|}{\left[\frac{\ell}{\ell+\lambda(k+p)}\right]^{m+1}\{(k+p)(1+B)+(A-B)(p-\alpha)\}}(k\in N) \quad \textbf{(4.13)}$$

by Corollary 1, setting

$$\mu_k=\frac{\left[\frac{\ell}{\ell+\lambda(k+p)}\right]^{m+1}\{(k+p)(1+B)+(A-B)(p-\alpha)\}}{1/\ell(A-B)(p-\alpha)|a_{-p}|}a_k\qquad(k\in N) \quad \textbf{(4.14)}$$

and

$$\mu_0=1-\sum_{k=1}^{\infty}\mu_k f_k(z), \quad \textbf{(2.15)}$$

it follows that

$$f(z)=\sum_{k=0}^{\infty}\mu_k f_k(z).$$

This completes the proof of Theorem6.

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5. INTEGRAL OPERATORS

Theorem 8: If $f(z)=\frac{a_{-p}}{z^p}+\sum_{k=1}^{\infty}(-1)^{k-1}a_k z^k$ $(a_{-p}\neq 0; a_k\geq 0, p\in N)$ is in the class $\sum_a^*(p,m,\alpha,\lambda,\ell,A,B)$, then

$$F(z)=(c-p+1)z^{-c-1}\int_0^z t^c f(t)dt$$

$$=\frac{a_{-p}}{z^p}+\sum_{k=1}^{\infty}(-1)^{k-1}\left(\frac{c-p+1}{c+k+1}\right)a_k z^k, \quad \textbf{(5.1)}$$

$c>p-1$, belongs to the class $\sum_a^*(p,m,\gamma,(p,\alpha,c,A,B),\lambda,\ell,A,B)$, where

$$\gamma(p,\alpha,c,A,B)=p-\frac{(c-p+1)(1+p)(1+B)(p-\alpha)}{(c+p+1)[(1+p)(1+B)+(A-B)(p-\alpha)]-(c-p+1)(A-B)(p-\alpha)}. \quad \textbf{(5.2)}$$

The result is sharp for the function $f(z)$ given by

$$f(z)=\frac{a_{-p}}{z^p}+\frac{(B-A)(p-\alpha)|a_{-p}|}{\left[\frac{\ell}{\ell+\lambda(1+p)}\right]^{m+1}\left[(1+p)(1+B)+(A-B)(p-\alpha)\right]}z. \quad \textbf{(5.3)}$$

Proof:
Suppose

$$f(z)=\frac{a_{-p}}{z^p}+\sum_{k=1}^{\infty}(-1)^{k-1}a_k z^k\in\sum_a^*(p,m,\alpha,\lambda,\ell,A,B),$$

in view of Theorem 2, we shall find the largest value of γ for which

$$\sum_{k=1}^{\infty}\frac{\left[\frac{\ell}{\ell+\lambda(k+p)}\right]^{m+1}\left[(k+p)(1+B)+(A-B)(p-\gamma)\right]}{(B-A)(p-\alpha)|a_{-p}|}\left(\frac{c-p+1}{c+k+1}\right)a_k\leq 1.$$

It is sufficient to fined the range values of γ for which

$$\frac{(c-p+1)\left[(k+p)(1+B)+(A-B)(p-\gamma)\right]}{(c+k+1)(p-\gamma)}\leq\frac{\left[(k+p)(1+B)+(A-B)(p-\alpha)\right]}{(p-\alpha)} \text{ for each } k.$$

Solving the above inequality for γ, we obtain

$$\gamma\leq p-\frac{(c-p+1)(k+p)(1+B)(p-\alpha)}{(c+k+1)\left[(k+p)(1+B)+(A-B)(p-\alpha)\right]-(c-p+1)(A-B)(p-\alpha)}.$$

For each $p,\alpha,\lambda,\ell,A,B$ and c fixed let

$$F(k)=p-\frac{(c-p+1)(k+p)(1+B)(p-\alpha)}{(c+k+1)\left[(k+p)(1+B)+(A-B)(p-\alpha)\right]-(c-p+1)(A-B)(p-\alpha)}.$$

Then $F(k+1)-F(k)=\frac{D}{G}>0$ for each k, where

$$D=(c-p+1)(1+B^2)(p-\alpha)(k+p)(k+p+1)$$

and

$$G=\{(c+k+2)[(k+p+1)(1+B)+(A-B)(p-\alpha)]-(c-p+1)(A-B)(p-\alpha)\}.$$

$$.\{(c+k+1)[(k+p)(1+B)+(A-B)(p-\alpha)]-(c-p+1)(A-B)(p-\alpha)\}.$$

Hence $F(k)$ is an increasing function of k. Since

$$F(1)=p-\frac{(c-p+1)(1+p)(1+B)(p-\alpha)}{(c+2)\left[(1+p)(1+B)+(A-B)(p-\alpha)\right]-(c-p+1)(A-B)(p-\alpha)},$$

the result follows.

Remark:

(i) Putting $a_{-p}=1, p=1, m=0, A=-1$ and $B=1$ in the above results, we have the results obtained by Uralegaddi and Ganigi [9];

(ii) Putting $a_{-p}=1,\ p=1,\ A=-1$ and $B=1$ in the above results, we have the results obtained by Aouf and Darwish [10];

(iii) Putting $a_{-p}=1$ and $p=1$ in the above results, we have the results obtained by Aouf et. al. [11].

6. CONVOLUTION PROPERTIES

Theorem 9:
If $f(z)$ and $g(z)$ belong to the clas $B_m(p,\alpha,\lambda,\ell,A,B)$, then

$$T(z)=\frac{a_{-p}b_{-p}}{z^p}+\sum_{k=1}^{\infty}(a_k^2+b_k^2)z^k, \quad \textbf{(6.1)}$$

is in the class $B_m(p,\alpha,\lambda,\ell,A,B)$ such that $A_1<-\mu^2+B_1(1-\mu^2)$, where

$$\mu=\frac{\sqrt{2(p-\alpha)(k+p)(|a_{-p}||b_{-p}|)}(B-A)}{\sqrt{2|a_{-p}||b_{-p}|(B-A)(p-\alpha)+\sqrt{\left[\frac{\ell}{\ell+\lambda(k+p)}\right]^{m+1}}\left[(k+p)(1+B)+(A-B)(p-\alpha)\right]}}.$$

Proof:
Since $f,g\in B_m(p,\alpha,\lambda,\ell,A,B)$. Theorem 2 yields

$$\sum_{k=1}^{\infty}\left(\frac{\left[\frac{\ell}{\ell+\lambda(k+p)}\right]^{m+1}\left[(k+p)(1+B)+(A-B)(p-\alpha)\right]a_k}{(B-A)(p-\alpha)|a_{-p}|}\right)^2\le 1,$$

and

$$\sum_{k=1}^{\infty}\left(\frac{\left[\frac{\ell}{\ell+\lambda(k+p)}\right]^{m+1}\left[(k+p)(1+B)+(A-B)(p-\alpha)\right]b_k}{(B-A)(p-\alpha)|b_{-p}|}\right)^2\le 1,$$

we obtain from the last two inequalities

$$\sum_{k=1}^{\infty}\frac{1}{2}\left(\frac{\left[\frac{\ell}{\ell+\lambda(k+p)}\right]^{m+1}\left[(k+p)(1+B)+(A-B)(p-\alpha)\right]}{(B-A)(p-\alpha)|a_{-p}||b_{-p}|}\right)^2(a_k^2+b_k^2)\le 1, \quad \textbf{(6.2)}$$

However $T(z)\in B_m(p,\alpha,\lambda,\ell,A,B)$ if and only if

$$\sum_{k=1}^{\infty}\left(\frac{\left[\frac{\ell}{\ell+\lambda(k+p)}\right]^{m+1}\left[(k+p)(1+B_1)+(A_1-B_1)(p-\alpha)\right]}{(B_1-A_1)(p-\alpha)|a_{-p}||b_{-p}|}\right)(a_k^2+b_k^2)\le 1. \quad \textbf{(6.3)}$$

where $-1\le A_1<B_1\le 1$, $\lambda\ge 0$, $\ell>0$, but (6.2) implies (6.3) if

$$\frac{\left[\frac{\ell}{\ell+\lambda(k+p)}\right]^{m+1}\left[(k+p)(1+B_1)+(A_1-B_1)(p-\alpha)\right]}{(B_1-A_1)(p-\alpha)|a_{-p}||b_{-p}|}$$

$$<\frac{1}{2}\left(\frac{\left[\frac{\ell}{\ell+\lambda(k+p)}\right]^{m+1}\left[(k+p)(1+B)+(A-B)(p-\alpha)\right]}{(B-A)(p-\alpha)|a_{-p}||b_{-p}|}\right)^2.$$

Hence, if

$$\frac{1+B_1}{B_1-A_1}<\frac{2(B-A)^2(p-\alpha)^2|a_{-p}||b_{-p}|+\left[\frac{\ell}{\ell+\lambda(k+p)}\right]^{m+1}\left[(k+p)(1+B)+(A-B)(p-\alpha)\right]^2}{2(B-A)^2(p-\alpha)(k+p)|a_{-p}||b_{-p}|}.$$

This is equivalent to

$$\frac{B_1-A_1}{1+B_1}>\frac{2(B-A)^2(p-\alpha)(k+p)|a_{-p}||b_{-p}|}{2(B-A)^2(p-\alpha)^2|a_{-p}||b_{-p}|+\left[\frac{\ell}{\ell+\lambda(k+p)}\right]^{m+1}\left[(k+p)(1+B)+(A-B)(p-\alpha)\right]^2}$$

$$=\mu^2. \quad \textbf{(6.4)}$$

Hence we get $A_1<-\mu^2+B_1(1-\mu^2)$.

Theorem 10: Let $f(z)$ and $g(z)$ belong to the class $B_m(p,\alpha,\lambda,\ell,A,B)$. Then the convolution (or Hadamard product) of two functions f and g belong to the class that is, $(f*g)(z)\in B_m(p,\alpha,\lambda,\ell,A,B)$, where $A_1<-v+B_1(1-v)$ and

$$v=\frac{(B-A)^2(p-\alpha)|a_{-p}||b_{-p}|}{(B-A)^2(p-\alpha)^2|a_{-p}||b_{-p}|+\left[\frac{\ell}{\ell+\lambda(k+p)}\right]^{m+1}\left[(k+p)(1+B)+(A-B)(p-\alpha)\right]^2}.$$

Proof:
Since $f,g\in B_m(p,\alpha,\lambda,\ell,A,B)$, by using the Cauchy-Schwarz inequality and Theorem2, we obtain

$$\sum_{k=1}^{\infty}\frac{\left[\frac{\ell}{\ell+\lambda(k+p)}\right]^{m+1}\left[(k+p)(1+B)+(A-B)(p-\alpha)\right]}{(B-A)(p-\alpha)|a_{-p}||b_{-p}|}$$

$$\le\left(\sum_{k=1}^{\infty}\frac{\left[\frac{\ell}{\ell+\lambda(k+p)}\right]^{m+1}\left[(k+p)(1+B)+(A-B)(p-\alpha)\right]}{(B-A)(p-\alpha)|a_{-p}|}a_k\right)^{1/2}\left(\sum_{k=1}^{\infty}\frac{\left[\frac{\ell}{\ell+\lambda(k+p)}\right]^{m+1}\left[(k+p)(1+B)+(A-B)(p-\alpha)\right]}{(B-A)(p-\alpha)|b_{-p}|}b_k\right)^{1/2}\le 1, \quad \textbf{(6.5)}$$

we must fined the values of A_1, B_1 so that

$$\sum_{k=1}^{\infty} \frac{\left[\frac{\ell}{\ell+\lambda(k+p)}\right]^{m+1}\left[(k+p)(1+B_1)+(A_1-B_1)(p-\alpha)\right]}{(B_1-A_1)(p-\alpha)\left|a_{-p}\right|\left|b_{-p}\right|} a_k b_k < 1. \quad \textbf{(6.6)}$$

There for, by (6.5), (6.6) holds true if

$$\sqrt{a_k b_k} \le \frac{(B_1-A_1)\left[(k+p)(1+B)+(A-B)(p-\alpha)\right]}{(B-A)\left[(k+p)(1+B_1)+(A_1-B_1)(p-\alpha)\right]}, \; k \ge m, \; m \ge p, \; a_k \ne 0, \; b_k \ne 0. \quad \textbf{(6.7)}$$

By (6.5), we have

$$\sqrt{a_k b_k} < \frac{(B-A)(p-\alpha)\left|a_{-p}\right|}{\left[\frac{\ell}{\ell+\lambda(k+p)}\right]^{m+1}\left[(k+p)(1+B)+(A-B)(p-\alpha)\right]},$$ therefor (6.7) holds true

if

$$\frac{\left[\frac{\ell}{\ell+\lambda(k+p)}\right]^{m+1}\left[(k+p)(1+B_1)+(A_1-B_1)(p-\alpha)\right]}{(B_1-A_1)(p-\alpha)\left|a_{-p}\right|\left|b_{-p}\right|} \le \left[\frac{\left[\frac{\ell}{\ell+\lambda(k+p)}\right]^{m+1}\left[(k+p)(1+B)+(A-B)(p-\alpha)\right]}{(B-A)(p-\alpha)\left|a_{-p}\right|\left|b_{-p}\right|}\right]^2,$$

which is equivalent to

$$\frac{1+B_1}{B_1-A_1} < \frac{(B-A)^2(p-\alpha)^2\left|a_{-p}\right|\left|b_{-p}\right| + \left[\frac{\ell}{\ell+\lambda(k+p)}\right]^{m+1}\left[(k+p)(1+B)+(A-B)(p-\alpha)\right]^2}{(B-A)^2(p-\alpha)\left|a_{-p}\right|\left|b_{-p}\right|}.$$

Alternatively, we can write

$$\frac{B_1-A_1}{1+B_1} > \frac{(B-A)^2(p-\alpha)\left|a_{-p}\right|\left|b_{-p}\right|}{(B-A)^2(p-\alpha)^2\left|a_{-p}\right|\left|b_{-p}\right| + \left[\frac{\ell}{\ell+\lambda(k+p)}\right]^{m+1}\left[(k+p)(1+B)+(A-B)(p-\alpha)\right]^2} = v.$$

Hence we get $A_1 < -v + B_1(1-v)$.

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