Accelerating Bianchi Type-I Universe with Time Varying G and Λ-term in General Relativity

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***Abstract*—The physical behaviour of cosmological solutions to Einstein's field equations for a spatially and anisotropic Bianchi type-I space-time has been investigated by taking into account the effect of time varying cosmological parameters viz., gravitational constant (G), cosmological constant (Λ) and deceleration parameter (DP) q. The concept of time dependent DP with some suitable assumption provides the average scale factor $a = \sinh^{\frac{1}{n}}(\alpha T)$, where n and α are positive constants. For $0 < n \leq 1$, this generates a class of accelerating models while for $n > 1$, the models exhibit phase transition from early decelerating phase to present accelerating phase which is in good agreement with recent astrophysical observations. The cosmological constant Λ is found to be a decreasing function of time and it approaches a small positive value at the present epoch which is corroborated by consequences from recent supernovae Ia observations. From recently developed Statefinder pair, the behaviour of different stages of the evolution of the universe has been studied. The physical and geometric significances of the cosmological models have also been discussed.**

***Index Terms*—Cosmology, Bianchi type-I universe, Variable cosmological parameters, ΛCDM model**
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I. INTRODUCTION

In Einsteinâs theory of gravity, the Newtonian gravitational constant G and the cosmological term Λ are considered to be fundamental constants. The Newtonian constant of gravitation G plays the role of a coupling constant between geometry of space and matter in Einsteinâs field equations. In an evolving universe, it is natural to take this constant as a function of time. Ever since Dirac [1] first considered the possibility of a variable G, there have been numerous modifications of general relativity to allow a variable G (Wesson [2]). Nevertheless, these theories have not gained wide acceptance. However, recently a modification (Berman [3]; Beesham [4]; Lau [5]; Abdel-Rahman [6]) was proposed in Einsteinâs field equations that treated G and Λ as coupling variables within the framework of general relativity. Canuto & Narlikar [7] showed that the G varying cosmology is consistent with whatever cosmological observations are available. Beesham [4]; Levitt [8]; Abdel-Rahman [9] discussed the possibility of an increasing G. The question of time variation is more interesting in the case of the gravitational coupling constant G since several theories of gravity have been suggested in which G decreases with cosmic time

The problem of the cosmological constant is salient yet unsettled in cosmology. The smallness of the effective cosmological constant recently observed ($\Lambda_0 \leq 10^{-56}\mathrm{cm}^{-2}$) poses the most difficult problems involving cosmology and elementary particle physics theory. To explain the striking cancellation between the "bare" cosmological constant and the ordinary vacuum energy contributions of the quantum fields, many mechanisms have been proposed during last few years [10]. The "cosmological constant problem" can be expressed as the discrepancy between the negligible value of Λ for the present universe as seen by the successes of Newton's theory of gravitation [11] whereas the values 10^{50} larger is expected by the Glashow-Salam-Weinberg model [12] and by grand unified theory (GUT) it should be 10^{107} larger [13]. The cosmological term Λ is then small at the present epoch simply because the universe is too old. A cosmological term Λ has been investigated from the aspects of fundamental physics, general relativity, and observations (Weinberg [10]; Garnavich et al. [14], Perlmutter et al. [15], [16], Riess et al. [17], [18], Schmidt et al. [19]). Since the present value Λ_0 is inferred to be very small from many observations (Alcaniz & Lima [20]; Freedman [21]), it is natural to consider that it has decreased from an extremely large value at the early epoch of the universe. It should also be very important to investigate the variability related to the quintessence (Zlatev, Wang, & Steinhardt [22]; Wang et al. [23]). However, there does not exist any reliable theory to explain the reason for the small value of Λ_0. Therefore, Λ has been examined in detail focusing on the variability as a function of cosmological quantities such as the scale factor (Chen & Wu [24]; Waga [25]; Arbab & Abdel-Rahman [26]). Based on phenomenological approaches, some efforts have continued to relate variable Λ models to the observations: distant Type Ia supernovae (Vishwakarma [27]), and several kinds of observations (Sahni [28]).

In recent past, Lau [29] and Lau and Prokhovnik [30] have suggested a generalized theory of gravitation comprising time-dependent G and Λ terms. Thus a modified theory was proposed addressing G and Λ as coupling scalars within the Einstein equations and hence the principle of equivalence necessitates that only metric tensor g_{ij} and not G and Λ must enter the equation of motion of particles and photons. In this way the usual conservation law holds. Taking the divergence of the Einstein equations and using the Bianchi identities we obtain an equation that moderates the variation of G and Λ. These are modified field equations that admit to take into account a variable G and Λ. Nevertheless this approach has some drawbacks, for example, it can not be derived from Hamiltonian, although there are several advantages in this approach. In the last few years, a number of works with variable G and Λ have been done in the framework of Einstein theory [31]−[50].

The simplest of anisotropic models are Bianchi type-I models whose spatial sections are flat but the expansion or contraction rates ate directional dependent. Recently, Pradhan et al. [51] obtained cosmological models in Bianchi type-I space-time by considering variable deceleration parameter. Motivated by the above discussions, in this paper, we have investigated Bianchi type-I cosmological models with time-dependent deceleration parameter with variable gravitational and cosmological "constants". The outline of the paper is as follows: In Sect. 2, the metric and the field equations are described. Section 3 deals with the quadrature solutions of field equations. In Sect. 4, the physical and geometric aspects of the models have been discussed. Section 5 deals with Statefinder diagnostic. Finally, conclusions are summarized in the last Sect. 6.

II. THE METRIC AND FIELD EQUATIONS

We consider the space-time metric of the spatially homogeneous and anisotropic Bianchi type-I of the form

$$ds^2 = -dt^2 + A^2(t)dx^2 + B^2(t)dy^2 + C^2(t)dz^2. \quad (1)$$

where A, B and C are the directional scale factors, which are functions of the cosmic time t. In general relativity, the Bianchi identities for the Einstein tensor G_{ij} and the vanishing covariant divergence of the energy momentum tensor T_{ij} together with imply that the cosmological term Λ is constant. In theories with a variable Λ-term, one either introduces new terms (involving scalar fields, for instance) into the left-hand-side of the Einstein's field equations to cancel the non-zero divergence of Λg_{ij} (Bergmann [52]; Wagoner [53] or interprets Λ as a matter source and moves it to the right-hand-side of the field equation (Zeldovich [54]), in which case energy momentum conservation is understood to mean

$$T_{ij} - \left(\frac{\Lambda}{8\pi G}\right) g_{ij} = 0. \quad (2)$$

Of course, the two approaches are equivalent for a given theory (Overduin [55]). Here we follow the later approach and assume that the cosmic matter is represented by the energy momentum tensor of perfect fluid augmented with the Λ-term as

$$T_{ij} = (\rho + p)u_i u_j + \left(p - \frac{\Lambda}{8\pi G}\right) g_{ij}, \quad (3)$$

together with a perfect gas equation of state

$$p = \gamma\rho, \quad 0 \leq \gamma \leq 1, \quad (4)$$

where ρ, p are the energy density, thermodynamical pressure and $u^i = (0, 0, 0, 1)$ is the four-velocity vector of the fluid comporting the relation

$$u_i u^i = -1. \quad (5)$$

The Einstein's field equations read as

$$R_{ij} - \frac{1}{2}Rg_{ij} = 8\pi G T_{ij}, \quad (6)$$

where R_{ij} is the Ricci tensor; $R = g^{ij}R_{ij}$ is the Ricci scalar, G is the gravitational constant.

In the field equations ((6) with (3), Λ accounts for vacuum energy with its energy density ρ_ν and p_ν satisfying the equation of state

$$p_\nu = -\rho_\nu = -\frac{\Lambda}{8\pi G} \quad (7)$$

The critical density and the density parameters for energy density and cosmological constant are, respectively, defined as

$$\rho_c = \frac{3H^2}{8\pi G} \quad (8)$$

$$\Omega_M = \frac{\rho}{\rho_c} = \frac{8\pi G\rho}{3H^2} \quad (9)$$

$$\Omega_\Lambda = \frac{\rho_\nu}{\rho_c} = \frac{\Lambda}{3H^2}, \quad (10)$$

where H is the mean Hubble parameter defined by (20) for metric (1). We observe that the density parameters Ω_M and Ω_Λ are singular when H = 0.

In a comoving system of coordinates, the field equations (6) with (3) for the metric (1) lead to the following set of independent equations:

$$\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\dot{A}\dot{B}}{AB} = -8\pi G(t)p + \Lambda(t), \quad (11)$$

$$\frac{\ddot{A}}{A} + \frac{\ddot{C}}{C} + \frac{\dot{A}\dot{C}}{AC} = -8\pi G(t)p + \Lambda(t), \quad (12)$$

$$\frac{\ddot{B}}{B} + \frac{\ddot{C}}{C} + \frac{\dot{B}\dot{C}}{BC} = -8\pi G(t)p + \Lambda(t), \quad (13)$$

$$\frac{\dot{A}\dot{B}}{AB} + \frac{\dot{B}\dot{C}}{BC} + \frac{\dot{C}\dot{A}}{CA} = 8\pi G\rho + \Lambda(t). \quad (14)$$

The covariant divergence of the (6) yields

$$\dot{\rho} + 3(\rho + p)H + \rho\frac{\dot{G}}{G} + \frac{\dot{\Lambda}}{8\pi G} = 0. \quad (15)$$

The usual energy conservation equation $T^{ij}_{;j} = 0.$ leads to

$$\dot{\rho} + 3(\rho + p)H = 0. \quad (16)$$

Now (15) reduces to

$$8\pi\dot{G}\rho + \dot{\Lambda} = 0. \quad (17)$$

The spatial volume is defined as

$$V^3 = ABC \quad (18)$$

We define the average scale factor a of anisotropic model as

$$a = (ABC)^{\frac{1}{3}} \quad (19)$$

So that the generalized mean Hubble parameter H is given by

$$H = \frac{\dot{a}}{a} = \frac{1}{3}\left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right) = \frac{1}{3}(H_x + H_y + H_z), \quad (20)$$

where $H_x = \frac{\dot{A}}{A}, H_y = \frac{\dot{B}}{B}, H_z = \frac{\dot{C}}{C}$ are the directional Hubble parameters in the direction of x, y and z respectively and a dot denotes differentiation with respect to cosmic time t.

The expression for the dynamical scalars such as the expansion scalar (θ), anisotropy parameter A_m and the shear scalar (σ) are defined as usual:

$$\theta = u^i_{;i} = \left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right), \tag{21}$$

$$A_m = \frac{1}{3}\sum_{i=1}^{3}\left(\frac{H_i - H}{H}\right)^2, \tag{22}$$

$$\sigma^2 = \frac{1}{2}\sigma_{ij}\sigma^{ij} = \frac{1}{2}\left[\left(\frac{\dot{A}}{A}\right)^2 + \left(\frac{\dot{B}}{B}\right)^2 + \left(\frac{\dot{C}}{C}\right)^2\right] - \frac{\theta^2}{6}, \tag{23}$$

where

$$\sigma_{ij} = \frac{1}{2}\left(u_{i;\alpha}P^{\alpha}_j + u_{j;\alpha}P^{\alpha}_i\right) - \frac{1}{3}\theta P_{ij}. \tag{24}$$

Here the projection tensor P_{ij} has the form

$$P_{ij} = g_{ij} - u_i u_j. \tag{25}$$

III. QUADRATURE SOLUTIONS OF THE FIELD EQUATIONS

The field equations (11)−(14) are a system of four equations with seven unknown parameters A, B, C, G, p, ρ and Λ. Three additional constraints relating these parameters are required to obtain explicit solution of the system.

We firstly assume the power law relation between he Gravitational Constant (G) and scale factor a as

$$G \propto a^m \tag{26}$$

where m is a constant. For sake of mathematical simplicity, (26) may be written as

$$G = G_0 a^m \tag{27}$$

where G_0 is a positive constant. Such type of relation has been already proposed in the literature [45, 50−54].

If we know that the value of scale factor a, we can solve the equations. We define the DP (deceleration parameter) q as

$$q = -\frac{a\ddot{a}}{\dot{a}^2} = -\left(\frac{\dot{H} + H^2}{H^2}\right) = b(t) \ \text{ say} \tag{28}$$

The motivation to choose such time dependent DP is behind the fact that the universe is accelerated expansion at present as observed in recent observations of Type Ia supernova [14−19]. Also, the transition redshift from deceleration expansion to accelerated expansion is about 0.5. Now for a Universe which was decelerating in past and accelerating at the present time, the DP must show signature flipping (Padmanabhan and Roychowdhury [56], Amendola [57]) between positive and negative values. So, in general, the DP is not a constant but time variable.

The Eq. (28) may be rewritten as

$$\frac{\ddot{a}}{a} + b\frac{\dot{a}^2}{a^2} = 0. \tag{29}$$

In order to solve the Eq. (29), we assume $b = b(a)$. It is important to note here that one can assume $b = b(t) = b(a(t))$, as a is also a time dependent function. It can be done only if there is a one to one correspondences between t and a. But this is only possible when one avoid singularity like big bang or big rip because both t and a are increasing function.

The general solution of Eq. (29) with assumption $b = b(a)$, is given by

$$\int e^{\int \frac{b}{a}da} da = t + k, \tag{30}$$

where k is an integrating constant.

One can not solve Eq. (30) in general as b is variable. So, in order to solve the problem completely, we have to choose $\int \frac{b}{a}da$ in such a manner that Eq. (30) be integrable without any loss of generality. Hence we consider

$$\int \frac{b}{a}da = \ln L(a). \tag{31}$$

which does not affect the nature of generality of solution. Hence from Eqs. (28) and (29), we obtain

$$\int L(a)da = t + k. \tag{32}$$

Of course the choice of $L(a)$, in Eq. (32), is quite arbitrary but, since we are looking for physically viable models of the universe consistent with observations, we consider

$$L(a) = \frac{na^{n-1}}{\alpha\sqrt{1 + a^{2n}}}, \tag{33}$$

where α is an arbitrary constant. In this case, on integrating, Eq. (32) with (33) gives an exact solution as

$$a = \sinh^{\frac{1}{n}}(\alpha T) \tag{34}$$

where $T = t + k$. The relation (34) is recently used by Pradhan [58], Pradhan et al. [59], Chanchal et al. [45] in studying cosmological models in different contexts where the authors have choosen the integrating constant k as zero. This relation (34) generalizes the value of scale factor obtained by Pradhan et al. [60], [61] in connection with the study of dark energy models in Bianchi type-VI_0 and cosmological models with variable Λ-term in Bianchi type-I space-time. We also note that $T = 0$ and $T = \infty$ respectively correspond to the proper time $t = -k$ and $t = \infty$.

Subtracting Eq. (11) from (12), (11) from (13), (12) from (13) and taking second integral of each, we get the following three relations

$$\frac{A}{B} = d_1 \exp\left(k_1 \int a^{-3}dt\right) \tag{35}$$

$$\frac{B}{C} = d_2 \exp\left(k_2 \int a^{-3}dt\right) \tag{36}$$

$$\frac{C}{A} = d_3 \exp\left(k_3 \int a^{-3}dt\right) \tag{37}$$

respectively, where d_1, d_2, d_3 and k_1, k_2, k_3 are constants of integration. Finally, using $a = (ABC)^{\frac{1}{3}}$, we write the metric functions from (35)−(37) in explicit form as

$$A(t) = l_1 a \exp\left(m_1 \int a^{-3} dt\right), \tag{38}$$

$$B(t) = l_2 a \exp\left(m_2 \int a^{-3} dt\right), \tag{39}$$

$$C(t) = l_3 a \exp\left(m_3 \int a^{-3} dt\right), \tag{40}$$

where

$$l_1 = \sqrt[3]{d_1 d_2}, \quad l_2 = \sqrt[3]{d_1^{-1} d_3}, \quad l_3 = \sqrt[3]{(d_2 d_3)^{-1}}, \tag{41}$$

and

$$m_1 = \frac{k_1 + k_2}{3}, \quad m_2 = \frac{k_3 - k_1}{3}, \quad m_3 = \frac{-(k_2 + k_3)}{3}, \tag{42}$$

The constants m_1, m_2, m_3 and l_1, l_2, l_3 satisfy the fallowing two relations:

$$m_1 + m_2 + m_3 = 0, \quad l_1 l_2 l_3 = 1. \tag{43}$$

Using (34), in Eqs. (38)−(40), we get

$$A(t) = l_1 \sinh^{\frac{1}{n}}(\alpha T) \exp\left[m_1 F(t)\right], \tag{44}$$

$$B(t) = l_2 \sinh^{\frac{1}{n}}(\alpha T) \exp\left[m_2 F(t)\right], \tag{45}$$

$$C(t) = l_3 \sinh^{\frac{1}{n}}(\alpha T) \exp\left[m_3 F(t)\right]. \tag{46}$$

where

$$F(t) = \int \sinh^{-\frac{3}{n}}(\alpha T) dT = \frac{1}{\alpha} \cosh(\alpha T) \{\sinh(\alpha T)\}^{1-\frac{3}{n}}$$

$$-\{\sinh^2(\alpha T)\}^{\frac{3}{n}-1}$$

$$Hypergeometric2F1\left[\frac{1}{2}, \frac{1}{2}\left(1 + \frac{3}{n}\right), \frac{3}{2}, \cosh^2(\alpha T)\right]$$

$$= 1 + \frac{1}{6}\left(1 + \frac{3}{n}\right)\cosh^2(\alpha T) +$$

$$\frac{3}{40}\left(1 + \frac{3}{n}\right)\left(1 + \frac{1}{n}\right)\cosh^4(\alpha T) + O[\cosh(\alpha T)]^6. \tag{47}$$

Hence the geometry of the universe (1) is reduced to

$$ds^2 = -dt^2 + \sinh^{\frac{2}{n}}(\alpha T)\Big[l_1^2 \exp\{2m_1 F(t)\} dx^2 +$$

$$l_2^2 \exp\{2m_2 F(t)\} dy^2 + l_3^2 \exp\{2m_3 F(t)\} dz^2\Big] \tag{48}$$

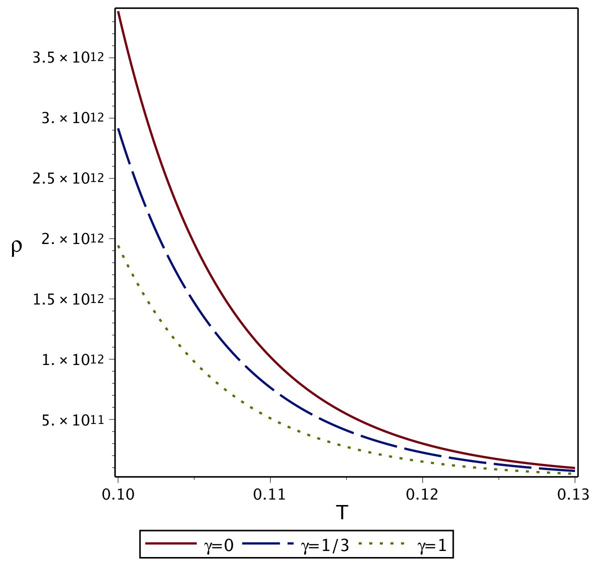

Figure 1. The plot of energy density ρ versus T. Here $\alpha = 1$, $\ell_0 = 1$

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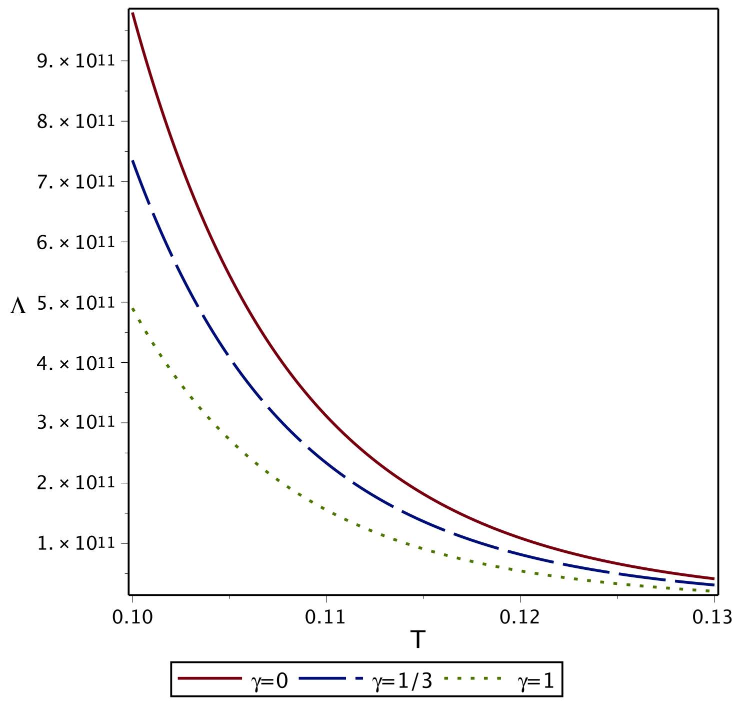

Figure 2. The plot of cosmological constant Λ versus T. Here $\alpha = 1$, $\ell_0 = 1$

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IV. SOME PHYSICAL AND GEOMETRIC PROPERTIES OF SOLUTIONS

From Eqs. (44)−(46), we obtain

$$\frac{\dot{A}}{A} = \frac{\alpha}{n}\coth(\alpha T) + m_1 \sinh^{-\frac{3}{n}}(\alpha T),$$

$$\frac{\dot{B}}{B} = \frac{\alpha}{n}\coth(\alpha T) + m_2 \sinh^{-\frac{3}{n}}(\alpha T),$$

$$\frac{\dot{C}}{C} = \frac{\alpha}{n}\coth(\alpha T) + m_3 \sinh^{-\frac{3}{n}}(\alpha T), \tag{49}$$

and

$$\frac{\ddot{A}}{A} = \frac{\alpha^2}{n} - \frac{\alpha b_1}{n}\coth(\alpha T)\{\sinh(\alpha T)\}^{-\frac{3}{n}} + b_1^2\{\sinh(\alpha T)\}^{\frac{-6}{n}},$$

$$\frac{\ddot{B}}{B} = \frac{\alpha^2}{n} - \frac{\alpha b_2}{n}\coth(\alpha T)\{\sinh(\alpha T)\}^{-\frac{3}{n}} + b_2^2\{\sinh(\alpha T)\}^{\frac{-6}{n}},$$

$$\frac{\ddot{C}}{C} = \frac{\alpha^2}{n} - \frac{\alpha b_3}{n}\coth(\alpha T)\{\sinh(\alpha T)\}^{-\frac{3}{n}} + b_3^2\{\sinh(\alpha T)\}^{\frac{-6}{n}}, \quad (50)$$

Using Eq. (34) into (27),the gravitational constant is

$$G = G_0\{\sinh(\alpha T)\}^{\frac{m}{n}} \quad (51)$$

Using Eqs. (49)−(51) and (4) and solving the field equations (11)−(14), we get the expressions for the energy density (ρ), the pressure (p), and the cosmological constant (Λ) for the model (48) as

$$\rho = \frac{1}{8\pi G_0(1+\gamma)}\left[\frac{2\alpha^2}{n^2}csch^{\frac{m}{n}+2}(\alpha T) + \beta_1 csch^{\frac{(m+6)}{n}}(\alpha T)\right], \quad (52)$$

$$p = \frac{\gamma}{8\pi G_0(1+\gamma)}\left[\frac{2\alpha^2}{n^2}csch^{\frac{m}{n}+2}(\alpha T) + \beta_1 csch^{\frac{(m+6)}{n}}(\alpha T)\right], \quad (53)$$

$$\Lambda = \frac{1}{(1+\gamma)}\left[\frac{\alpha^2}{n^2}\{2+(1+3\gamma)\coth^2(\alpha T)\} + \beta_2 csch^{\frac{6}{n}}(\alpha T)\right], \quad (54)$$

where

$$\beta_1 = m_3(m_1+m_2) - (m_1^2+m_2^2),$$

$$\beta_2 = (m_1^2+m_2^2+m_1m_2) + \gamma(m_1m_2+m_2m_3+m_3m_1).$$

From above relations (52)−(54), we can obtain four types physically relevant models:

- When $\gamma = 0$, we obtain empty model.
- When $\gamma = \frac{1}{3}$, we obtain radiating dominated model.
- When $\gamma = -1$, we have the degenerate vacuum or false vacuum or ρ vacuum model (Cho [62])).
- When $\gamma = 1$, the fluid distribution corresponds with the equation of state $\rho = p$ which is known as Zeldovich fluid or stiff fluid model (Zeldovich [63])).

From Eq. (52), it is observed that the energy density ρ is a decreasing function of time and $\rho > 0$ always. The energy density has been graphed versus time T in Fig. 1 for $\gamma = 0, \frac{1}{3}, 1$. It is evident that the energy density remains positive in all three types of model. However, it decreases more sharply with the cosmic time in Zeldovich universe, compare to radiating dominated and empty fluid universes.

Figure 2 is the plots of cosmological term Λ versus time T for $\gamma = 0, \frac{1}{3}, 1$. In all three types of models, we observe that Λ is decreasing function of time t and it approaches a small positive value at late time (i.e. at present epoch). However, it decreases more sharply with the cosmic time in empty universe, compare to radiating dominated and stiff fluid universes. Such type of behaviour for Λ corresponds to repulsion with cosmic acceleration with is in good agreement with recent observations. Recent cosmological observations (Garnavich et al. [14]; Perlmutter et al. [15], [16]; Riess et al. [17], [18]; Schmidt et al. [19]) suggest the existence of a positive cosmological constant Λ with the magnitude $\Lambda(G\hbar/c^3) \approx 10^{-123}$. These observations on magnitude and red-shift of type Ia supernova suggest that our universe may be an accelerating one with induced cosmological density through the cosmological Λ-term. Thus, the nature of Λ in our derived model is supported by recent observations.

The vacuum energy density (ρ_ν), critical density (ρ_c) and the density parameters (Ω_M, Ω_Λ) read as

$$\rho_\nu = \frac{1}{8\pi G_0(1+\gamma)}\left[\frac{\alpha^2}{n^2}\{2+(1+3\gamma)\coth^2(\alpha T)\}csch^{\frac{m}{n}}(\alpha T) + \beta_3 csch^{\frac{(m+6)}{n}}(\alpha T)\right] \quad (55)$$

$$\rho_c = \frac{3\alpha^2}{8\pi G_0 n^2}\left[csch^{\frac{m}{n}+2)}(\alpha T)\cosh^2(\alpha T)\right] \quad (56)$$

$$\Omega_M = \frac{n^2\left[\frac{2\alpha^2}{n^2}csch^2(\alpha T) + \beta_1 csch^{\frac{6}{n}}(\alpha T)\right]}{3(1+\gamma)\alpha^2\coth^2(\alpha T)} \quad (57)$$

$$\Omega_\Lambda = \frac{n^2\left[\frac{\alpha^2}{n^2}\{2+(1+3\gamma)\coth^2(\alpha T)\} + \beta_2 csch^{\frac{6}{n}}(\alpha T)\right]}{3(1+\gamma)\alpha^2\coth^2(\alpha T)} \quad (58)$$

Adding (57) and (58), we get

$$\Omega_{total} = \Omega_M + \Omega_\Lambda = 1 + \frac{\beta n^2 csch^{\frac{6}{n}}(\alpha T)}{3(1+\gamma)\alpha^2\coth^2(\alpha T)} \quad (59)$$

where $\beta = (m_1m_2 + m_2m_3 + m_3m_1)$. For $\beta = 0$, we have

$$\Omega_{total} = 1, \quad (60)$$

which clearly describes the standard FRW model. We also observe that $\Omega_{total} \to 1$ as $T \to \infty$. Thus, we see that the total density parameter approaches to 1 for sufficiently large time for all the three models $\gamma = 0, \frac{1}{3}, 1$ which is reproducible with current observations.

The expressions for observational physical quantities such as spatial volume (V), directional Hubble parameters (H_i), Hubble parameter (H), expansion scalar (θ), shear scalar (σ) and the mean anisotropy parameter (A_m are obtained for the model (48) as:

$$V = \sinh(\alpha T)^{\frac{3}{n}}, \quad (61)$$

The Hubble parameter is

$$H_i = \frac{\alpha}{n}\coth(\alpha T) + m_i\ sech^{\frac{3}{n}}(\alpha T), \quad i = 1,2,3, \quad (62)$$

$$\theta = 3H = \frac{3\alpha}{n}\coth(\alpha T), \quad (63)$$

$$\sigma^2 = \frac{\beta_3}{2}\sinh^{-\frac{6}{n}}(\alpha T), \quad (64)$$

$$A_m = \frac{\beta_3}{3\alpha^2}\frac{csch^{\frac{6}{n}}(\alpha T)}{\coth^2(\alpha T)} \quad (65)$$

where

$$\beta_3 = m_1^2 + m_2^2 + m_3^2.$$

The deceleration parameter is given by

$$q = -1 + n\left[1 - \tanh^2(\alpha T)\right] \quad (66)$$

From Eqs. (61) and (63), we observe that the spatial volume is zero at $T = 0$ and the expansion scalar is infinite, which show that the universe starts evolving with zero

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volume at $T = 0$ which is big bang scenario. From Eqs. (44)−(46), we observe that the spatial scale factors are zero at the initial epoch $T = 0$ and hence the model has a point type singularity (MacCallum [64])). We observe that proper volume increases exponentially with time. Thus, the model represents the inflationary scenario. The directional Hubble parameters (H_i) are infinite at $T = 0$ and decays to zero as $T \to \infty$. They deviate from the mean Hubble parameter (H) due to the parameters m_1, m_2 and m_3 which obviously parametrize the difference between directional Hubble parameters. Since $m_1 + m_2 + m_3 = 0$, hence these parameters m_i also characterize the overall expansion of the universe. From Eqs. (63) and (64), we obtain

$$\frac{\sigma^2}{\theta^2} = \frac{\beta_3 n^2}{18\alpha^2 \coth^2(\alpha T) \sinh^{\frac{6}{n}}(\alpha T)}. \quad (67)$$

From above Eq. (67), we observe that $\frac{\sigma^2}{\theta^2}$ tends to zero as $T \to \infty$ indicating that the models approach to isotropy at late time i.e., at present epoch. The dynamics of the mean anisotropic parameter depends on the constant β_3. From Eq. (65), we observe that at late time when $t \to \infty$, $A_m \to 0$. Thus, our model has transition from initial anisotropy to isotropy at present epoch which is in good harmony with current observations. Thus, the observed isotropy of the universe can be achieved in our model at present epoch.

From Eq. (66), we observe that $q > 0$ for $T < \frac{1}{\alpha}\tanh^{-1}(1 - \frac{1}{n})^{\frac{1}{2}}$ and $q < 0$ for $T > \frac{1}{\alpha}\tanh^{-1}(1 - \frac{1}{n})^{\frac{1}{2}}$. It is also observed that for $0 < n \leq 1$, our model is in accelerating phase but for $n > 1$, our model is evolving from decelerating phase to accelerating phase. Also, recent observations of SNe Ia, expose that the present universe is accelerating and the value of DP lies to some place in the range $-1 \leq q < 0$. It follows that in our derived model, one can choose the value of DP consistent with the observations.

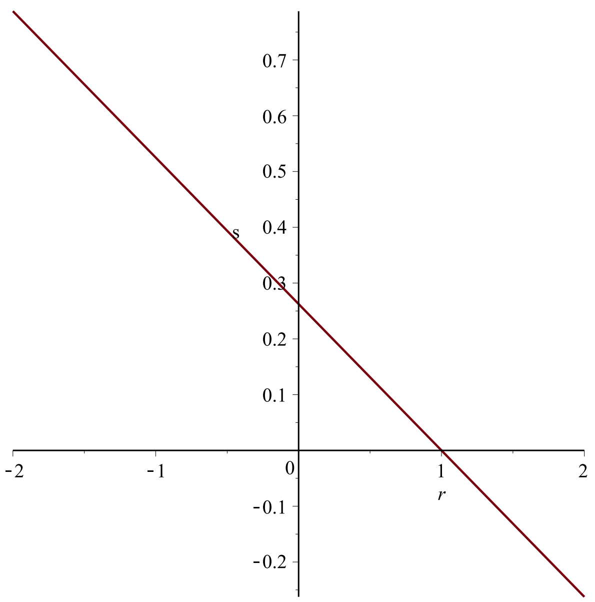

Figure 3. The plot of s against r.

V. STATEFINDER DIAGNOSTIC PARAMETERS

In this section, we calculate the statefinder parameters for the above three types of cosmological models. Since more and more dark energy models have been developed to explain the current cosmic acceleration, a method for discriminating contenders in a model independent manner was proposed by Sahni et al. [65] and Alam et al. [66]. They introduced a pair of cosmological diagnostic pair $\{r, s\}$ which they termed as Statefinder. The two parameters are dimensionless and are geometrical since they are derived from the cosmic scale factor alone. Over the past few decades, a simple cosmological model called the lambda cold dark-matter (ΛCDM) model has emerged as the best fit to the current observational data. ΛCDM stands for cosmological constant which is currently associated with a vacuum energy or dark energy inherent in empty space that explains the current expansion of space against the attractive (collapsing) effects of gravity. In order to explain the cosmic acceleration a form of negative-pressure matter called dark energy was suggested. ΛCDM has achieved several observations with outstanding success. For a current review, see Magaña et al. [67]).

The statefinder parameters can effectively differentiate between different form of dark energy and provide simple diagnosis regarding whether a particular model fits into the basic observational data. The above statefinder diagnostic pair has the following form:

$$r = 1 + 3\frac{\dot{H}}{H^2} + \frac{\ddot{H}}{H^3} \text{ and } s = \frac{r-1}{3(q-\frac{1}{2})}. \quad (68)$$

For the present models, the parameters $\{r, s\}$ can explicitly written in terms of T as

$$r = 1 + \frac{\alpha^2}{nH^2}\left[\alpha \coth(\alpha T)\left[\coth^2(\alpha T) + csch^2(\alpha T)\right] - 3csch^2(\alpha T)\right], \quad (69)$$

$$s = -\frac{2\alpha^2}{3nH^2(2q-1)}\left[3csch^2(\alpha T) - \alpha \coth(\alpha T)\left[\coth^2(\alpha T) + csch^2(\alpha T)\right]\right]. \quad (70)$$

Therefore, we can get the relation between r and s as

$$s = \frac{1}{3(q-\frac{1}{2})}(r-1). \quad (71)$$

Fig. 3 shows the variation of r against s using the current value of the deceleration parameter $q_0 = -0.77$. From this figure, we observe that s is negative when $r \geq 1$. We also observe that the universe starts from an Einstein static era $(r \to \infty, s \to -\infty)$ and goes to the ΛCDM model $(r = 1, s = 0)$.

VI. CONCLUSIONS

In this paper, exact solutions to Einstein's field equations for spatially homogeneous and anisotropic Bianchi type-I models with variable gravitational constant G and cosmological constant Λ have been obtained. To get the deterministic solution we choose a time-dependent deceleration parameter which yields a scale factor $a = \sinh^{\frac{1}{n}}(\alpha T)$, which represents two types of models: (i) the accelerating models for $0 < n \leq 1$ and (ii) the models with transition from the early decelerated phase to the present accelerating phase for $n > 1$ which is in good agreement with recent supernova Ia observations ((Garnavich et al. [14]; Perlmutter et al. [15], [16]; Riess et al. [17], [18]; Schmidt et al. [19]). It is also observed that the Λ-term is a decreasing function of time and it converges to a small positive value at late time for all three models $\gamma = 0, \frac{1}{3}, 1$. The nature of decaying vacuum energy density $\Lambda(T)$ in our derived models is supported by the recent cosmological observations of magnitude and redshift of SNe Ia, which suggest that our universe may be an accelerating one with induced cosmological density through the cosmological Λ-term. We also observe that the total density parameter (Ω_{total}) approaches 1 for sufficiently large time (i.e. at present epoch) for all the three models, i.e. matter dominated, radiating and stiff fluid of the universe which is in good agreement with current observations.

For different choice of n, we can generate a class of viable cosmological models of the universe in Bianchi type-I space-time. For example, if we set $n = 1$ in Eq. (34), we find $a = \sinh(\alpha T)$ which is used by Pradhan et al. [60] in studying the accelerating dark energy models in Bianchi type-VI_0 space-time and Pradhan et al. [61] in studying Bianchi type-I cosmological models with time dependent Λ-term. It is observed that such models are also in good harmony with current observations. If we put $n = 1$ in the results obtained in this paper, we obtain new cosmological models for scale factor $a = \sinh(\alpha T)$ in Bianchi type-I space-time.

$\{r, s\}$ diagram (Fig. 3) shows that the evolution of the universe starts from Einstein static era ($r \to \infty, s \to -\infty$) and approaches to ΛCDM model ($r = 1, s = 0$). So, from the Statefinder parameter $\{r, s\}$, the behaviour of different stages of the evolution of the universe have been generated. Our derived models are very close to ΛCDM model.

Thus, the solutions demonstrated in this paper may be useful for better understanding of the characteristic of Bianchi type-I cosmological models in the evolution of the universe within the framework of general theory of relativity.

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