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An Epidemic Model with an Asymptotically Homogeneous Transmission Function

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ABSTRACT

The rate of incidence plays an important role in epidemic model. In this context, an epidemic model with an asymptotically homogeneous transmission function was analyzed by Pathak et.al. [11]. They assumed that the entire interaction population becomes infectious. However, it is a little bit away from reality. In this paper, this assumption is modified by taking a part of this population as infected by introducing a new control parameter ρ. We have carried out the global qualitative analysis of an SIR epidemic model. By numerical simulation, it has been observed that increasing value of ρ increase I^* where as S^* and R^* decrease. The model is mathematically more general and is in full agreement with that of Pathak et.al.[11].

Keywords: *Epidemiology, stability, Biological equilibrium, Infectious diseases.*

1. INTRODUCTION

Transmissible diseases have tremendous influence in human life. Every year billions of people suffer or die in various infectious diseases. In recent times, the emerging and reemerging communicable diseases have led to a revive interest in the study of infectious diseases and become an important tool to analyze the spreading and controlling the diseases [6]. Epidemiology modeling and mathematical study on that model can contribute to the design and analysis of epidemiological surveys; suggest some crucial conclusions that should be collected. By determining the key parameters and finding its effects of changes in the parameter values, it identifies the trends, make general forecasts and finally estimate the uncertainty in forecasts.

Incidence rate plays an important role in the modeling of epidemic dynamics. The delay appears in incidence term which is typically the only nonlinearity, and is therefore the "cause" of all "interesting behavior". Various forms have been used for the incidence term, both for ODEs and for delay equations. After study of the cholera epidemic spread in Bari in 1973, Capasso et.al. [13] introduced a saturated incidence rate into epidemic models.

It has been suggested by many authors that the diseases transmission process may have a nonlinear incidence rate. Ruan et.al. [12] studied the global dynamics of an epidemic model with vital dynamics and nonlinear incidence rate of saturated mass action. Hethcote et.al. [7] proposed some epidemiological model with nonlinear incidence. Epidemiological model with nonlinear incidence rates $\beta I^p S^q$ were investigated by Liu et, al. [14] and Liu et, al. [15]. They performed a detailed analysis of co-dimension 1 bifurcation for the SEIRS and SIRS models with the incidence rate. Epidemiological models with nonlinear incidence rates $\lambda I^p S^q$ show a much wider range of dynamical behavior than do those with bilinear incidence rate λIS. These behavior are determined mainly by p and λ, and secondarily by q.

Many authors presented epidemiological model with time delay. Beretta et.al. [5] and Ma et.al. [16] Studied the global stability of a SIR epidemic model with time delay. Beretta et.al. [3] Studied the global asymptotic stability of a SIR epidemic model with distributed time delay. Ma et.al. [17] Considered the permanence of an SIR epidemic model with distributed time delay. Based on some known techniques on limit sets of differential dynamical system, they showed that, for any time delay, the SIR epidemic model is permanent if and only if an endemic equilibrium exists.

Diseases transmission is a dynamical process driven by the interaction of susceptible and infective. In this context, an SIRS model with an asymptotically homogeneous transmission function was analyzed by Pathak et.al. [11]. They assumed that the entire interaction population becomes infectious. However, it is a little bit away from reality. We have modified this assumption by taking a part of this population as infected by introducing a new control parameter. The model is mathematically more general thus includes the results of Pathak et.al. [11] as a special case.

2. THE BASIC MATHEMATICAL MODEL

The model we analyze in this paper is considered under the framework of the following nonlinear ordinary differential equations:

$$\begin{aligned}
\frac{dS}{dt} &= b - dS - \frac{kSI}{1+\alpha S+\beta I} + \gamma R, \\
\frac{dI}{dt} &= \frac{\rho kSI}{1+\alpha S+\beta I} - (d+\mu) I, \\
\frac{dR}{dt} &= \mu I - (d+\gamma) R + \frac{(1-\rho) kSI}{1+\alpha S+\beta I}.
\end{aligned} \tag{1}$$

Where $\rho \le 1$ is a positive constant and the other symbols have the same meaning as defined in [11].

3. EXISTENCE OF EQUILIBRIA

For the system (1), for any values of parameters, it always has a diseases-free equilibrium $E_0 = \left(\frac{b}{d}, 0, 0\right)$. Define the basic reproduction number as follows:

$$R_0 = \frac{\rho kb - b\alpha(d+\mu)}{d(d+\mu)}. \quad (2)$$

If $R_0 \le 1$, then there is no positive equilibrium. If $R_0 > 1$, then there is a unique positive equilibrium $E^*(S^*, I^*, R^*)$ of the system (1), called the "endemic equilibrium", given by

$$S^* = \frac{(d+\mu)(1+\beta I^*)}{\rho k - \alpha(d+\mu)},$$

$$I^* = \frac{(d+\gamma)\left[\rho kb - (b\alpha + d)(d+\mu)\right]}{\beta d(d+\mu)(d+\gamma) + \frac{d}{\rho}(d+\mu+\gamma)(\rho k - \alpha(d+\mu))},$$

$$R^* = \frac{1}{(d+\gamma)}\left[\mu I^* + \frac{(1-\rho)kS^*I^*}{1+\alpha S^* + \beta I^*}\right].$$

Now we check the dynamical behavior of diseases-free equilibrium point E_0 and endemic equilibrium point E^*. The variation matrix at E_0 is

$$V_0 = \begin{pmatrix} -d & \frac{kb}{(b\alpha + d)} & \gamma \\ 0 & \frac{\rho kb}{(b\alpha + d)} - (d+\mu) & 0 \\ 0 & \mu + \frac{(1-\rho)kb}{(b\alpha + d)} & -(d+\gamma) \end{pmatrix}$$

Its characteristic equation is given by

$$(-d-\lambda)\left(\frac{\rho kb}{(b\alpha + d)} - d - \mu - \lambda\right)(-d-\gamma-\lambda) = 0$$

Clearly, E_0 is stable if $\frac{\rho kb}{(b\alpha + d)} < (d+\mu)$.

The variation matrix at E^* is

$$V^* = \begin{pmatrix} -d - \frac{kI^*(1+\beta I^*)}{(1+\alpha S^* + \beta I^*)^2} & -\frac{kS^*(1+\alpha S^*)}{(1+\alpha S^* + \beta I^*)^2} & \gamma \\ \frac{\rho kI^*(1+\beta I^*)}{(1+\alpha S^* + \beta I^*)^2} & \frac{\rho kS^*(1+\alpha S^*)}{(1+\alpha S^* + \beta I^*)^2} - (d+\mu) & 0 \\ \frac{(1-\rho)kI^*(1+\beta I^*)}{(1+\alpha S^* + \beta I^*)^2} & \mu + \frac{(1-\rho)kS^*(1+\alpha S^*)}{(1+\alpha S^* + \beta I^*)^2} & -(d+\gamma) \end{pmatrix}$$

Its characteristic equation is given by

$$\lambda^3 + a_1\lambda^2 + b_1\lambda + c_1 = 0 \quad (3)$$

Where

$$a_1 = \left(3d + \mu + \gamma + \frac{kI^*(1+\beta I^*)}{(1+\alpha S^* + \beta I^*)^2} - \frac{\rho kS^*(1+\alpha S^*)}{(1+\alpha S^* + \beta I^*)^2}\right)$$

$$b_1 = \left(d + \frac{kI^*(1+\beta I^*)}{(1+\alpha S^* + \beta I^*)^2}\right)\left(2d + \mu + \gamma - \frac{\rho kS^*(1+\alpha S^*)}{(1+\alpha S^* + \beta I^*)^2}\right) + (d+\gamma)\left(d + \mu - \frac{\rho kS^*(1+\alpha S^*)}{(1+\alpha S^* + \beta I^*)^2}\right) + \frac{\rho k^2 S^* I^*(1+\alpha S^*)(1+\beta I^*)}{(1+\alpha S^* + \beta I^*)^4} - \frac{\gamma(1-\rho)kI^*(1+\beta I^*)}{(1+\alpha S^* + \beta I^*)^2}$$

$$c_1 = \left(d + \frac{kI^*(1+\beta I^*)}{(1+\alpha S^* + \beta I^*)^2}\right)\left(-\frac{\rho kS^*(1+\alpha S^*)}{(1+\alpha S^* + \beta I^*)^2} + (d+\mu)\right)(d+\gamma) - \left(\frac{\gamma\rho kI^*(1+\beta I^*)}{(1+\alpha S^* + \beta I^*)^2}\right)\left(\mu + \frac{(1-\rho)kS^*(1+\alpha S^*)}{(1+\alpha S^* + \beta I^*)^2}\right) - \frac{\gamma(1-\rho)kI^*(1+\beta I^*)}{(1+\alpha S^* + \beta I^*)^2}\left(-\frac{\rho kS^*(1+\alpha S^*)}{(1+\alpha S^* + \beta I^*)^2} + (d+\mu)\right) + \frac{\rho k^2 S^* I^*(1+\alpha S^*)(1+\beta I^*)(d+\gamma)}{(1+\alpha S^* + \beta I^*)^4}$$

Here $a_1 > 0, b_1 > 0$ and $c_1 > 0$ provided

$$\frac{\rho k S^*\left(1+\alpha S^*\right)}{\left(1+\alpha S^*+\beta I^*\right)^2}<(d+\mu),$$

Performing simple calculations it can easily be verified that $a_1 b_1 > c_1$ under the above conditions. Thus by Routh Hurwitz criterion, all Eigen values of (3) will have negative real part. Hence $E^*\left(S^*, I^*, R^*\right)$ is asymptotically stable.

4. MATHEMATICAL ANALYSIS

LEMMA: The plane $S+I+R=\frac{b}{d}$ is a manifold of system (1), which is attracting in the first octant.

PROOF: Summing up the three equations in (1) and denoting $N(t)=S(t)+I(t)+R(t)$, we have

$$\frac{dN}{dt}=b-dN. \qquad (4)$$

It is clear that $N(t)=\frac{b}{d}$ is a solution of (4) and for any $N(t_0)\geq 0$, the general solution of (4) is $N(t)=\frac{1}{d}\left[b-\left(b-dN\left(t_0\right)\right)e^{-d\left(t-t_0\right)}\right]$.

Thus, $\lim_{x\to\infty} N(t)=\frac{b}{d}$,

This implies the conclusion.

It is clear that the limit set of system (1) is on the plane $S+I+R=\frac{b}{d}$ .thus, we focus on the reduced system

$$\begin{aligned}\frac{dI}{dt}&=\frac{\rho dkI}{(b\alpha+d)+(\beta-\alpha)dI-\alpha dR}\left(\frac{b}{d}-I-R\right)-\\&\quad(d+\mu)I\equiv P(I,R),\\\frac{dR}{dt}&=\mu I-(d+\gamma)R+\\&\quad\frac{(1-\rho)dkI}{(b\alpha+d)+(\beta-\alpha)dI-\alpha dR}\left(\frac{b}{d}-I-R\right)\equiv Q(I,R).\end{aligned} \qquad (5)$$

THEOREM: System (5) does not have non-trivial periodic orbits if

$$(\beta-\alpha)(2d+\mu+\gamma)>2R\alpha(d+\gamma).$$

PROOF: Consider system (5) for I>0 and R>0. Take Dulac function [11] as

$$D(I,R)=\frac{(b\alpha+d)+(\beta-\alpha)dI-\alpha dR}{\rho dkI}$$

Then

$$\begin{aligned}\frac{\partial}{\partial I}(DP)+\frac{\partial}{\partial R}(DQ)&=-1-\frac{\mu\alpha}{\rho k}-\frac{(b\alpha+d)(d+\gamma)}{\rho dkI}-\frac{1-\rho}{\rho}-\\&\quad\frac{1}{\rho kI}\left[(\beta-\alpha)(2d+\mu+\gamma)I-2R\alpha(d+\gamma)\right]\\&<0\end{aligned}$$

If $(\beta-\alpha)(2d+\mu+\gamma)I>2R\alpha(d+\gamma).$

Hence, the conclusion follows.

In order to study the properties of the diseases-free equilibrium E_0 and the endemic equilibrium E^*, we rescale (5) by

$$x=\frac{k}{d+\gamma}I, y=\frac{k}{d+\gamma}R, \tau=(d+\gamma)t.$$

Then we obtain,

$$\begin{aligned}\frac{dx}{d\tau}&=\frac{px}{1+qx-ry}(A-x-y)-mx,\\\frac{dy}{d\tau}&=sx-y+\frac{Lx}{1+qx-ry}(A-x-y).\end{aligned} \qquad (6)$$

Where

$$p=\frac{\rho d}{d+\alpha b}, q=\frac{d(d+\gamma)(\beta-\alpha)}{k(d+\alpha b)}, r=\frac{\alpha d(d+\gamma)}{k(d+\alpha b)},$$

$$A=\frac{bk}{d(d+\gamma)}, m=\frac{(d+\mu)}{(d+\gamma)}, L=\frac{(1-\rho)d}{(d+\alpha b)}, s=\frac{\mu}{(d+\gamma)}.$$

The trivial equilibrium $(0,0)$ of the system (6) is the diseases-free equilibrium E_0 of model (1) and the unique positive equilibrium $\left(x^*, y^*\right)$ of system (6) is the endemic equilibrium E^* of model (1) where

$$x^*=\frac{Ap-m-(p-mr)y^*}{p+mq}$$

Which is positive if

$$(Ap-m)>(p-mr)y^*$$

and y^* is the positive solution of the following quadratic equation

$$a_2 y^2 + b_2 y + c_2 = 0 \qquad (7)$$

where

$$a_2 = \frac{r+q}{(p+mq)^2}\left[p^2(s+1) + m\{p(q-rs) + L(p-mr)\}\right],$$

$$b_3 = \frac{1}{(p+mq)^2}\left[(sp+mL)\{m(2r+q) - p(2Aq+1) - Ar(p-mq)\} - p(Aq+1)(p+mq)\right],$$

$$c_3 = \frac{1}{(p+mq)^2}(Aq+1)(Ap-m)(ps+mL).$$

Obviously, equation (7) has positive root if $b_2 < 0, b_2^2 - 4a_3c_3 > 0$. Thus the equilibrium point E^* exists.

We first determine the stability and topological type of $(0,0)$. The Jaccobian matrix of system (6) at (0,0) is

$$M_0 = \begin{pmatrix} Ap-m & 0 \\ LA+s & -1 \end{pmatrix}$$

If $Ap - m = 0$, then there exists a small neighborhood N_0 of $(0,0)$ such that the dynamics of system (6) are equivalent to that of

$$\frac{dx}{d\tau} = -px^2 + o\left((x,y)^2\right),$$

$$\frac{dy}{d\tau} = sx - y + \frac{Lx}{1+qx-ry}(A-x-y).$$

THEOREM: The diseases-free equilibrium (0,0) of system (6) is

i. A stable hyperbolic node if $A < \frac{m}{p}$;

ii. A saddle-node if $A = \frac{m}{p}$;

iii. A hyperbolic saddle if $A > \frac{m}{p}$.

When $A > \frac{m}{p}$., we discuss the stability and topological type of the endemic equilibrium (x^*, y^*).

The Jacobian of the system (6) at (x^*, y^*) is

$$M_1 = \begin{pmatrix} \frac{p\left[(1-ry^*)(A-2x^*-y^*) - qx^{*2}\right]}{(1+qx^*-ry^*)^2} - m & \frac{p\left[(Ar-1)x^* - (r+q)x^{*2}\right]}{(1+qx^*-ry^*)^2} \\ \frac{L\left[(1-ry^*)(A-2x^*-y^*) - qx^{*2}\right]}{(1+qx^*-ry^*)^2} + s & \frac{L\left[(Ar-1)x^* - (r+q)x^{*2}\right]}{(1+qx^*-ry^*)^2} - 1 \end{pmatrix}$$

The determinant of M_1 is

$$\det(M_1) = m - \frac{\left[(mL+ps)\{(Ar-1)x^* - (r+q)x^{*2}\}\right]}{(1+qx^*-ry^*)^2} + \frac{p\{(1-ry^*)(A-2x^*-y^*) - qx^{*2}\}}{(1+qx^*-ry^*)^2}.$$

Thus, $\det(M_1) > 0$ if

$$m > \frac{\left[(mL+ps)\{(Ar-1)x^* - (r+q)x^{*2}\}\right]}{(1+qx^*-ry^*)^2} + \frac{p\{(1-ry^*)(A-2x^*-y^*) - qx^{*2}\}}{(1+qx^*-ry^*)^2} \qquad (8)$$

Thus, if the above conditions hold then (x^*, y^*) is a node or a focus or a centre.

The trace of matrix M_1 is given by

$$tr(M_1) = \frac{p\left[(1-ry^*)(A-2x^*-y^*) - qx^{*2}\right]}{(1+qx^*-ry^*)^2} - m + \frac{L\left[(Ar-1)x^* - (r+q)x^{*2}\right]}{(1+qx^*-ry^*)^2} - 1$$

Thus, $tr(M_1) < 0$ if

$$(m+1) > \frac{p\left[(1-ry^*)(A-2x^*-y^*) - qx^{*2}\right]}{(1+qx^*-ry^*)^2} + \frac{L\left[(Ar-1)x^* - (r+q)x^{*2}\right]}{(1+qx^*-ry^*)^2} \qquad (9)$$

Thus, if both the conditions (8) and (9) are satisfied, we get a unique endemic equilibrium (x^*, y^*) of model (6), which is stable.

5. NUMERICAL SIMULATION

We take the parameters of the system as d=2.29, α=3.1, β=4.7, b=3.1, γ=1.5. k=9, μ=0.19, then $E_0 = (1.3537, 0, 0)$ and $0 < R_0 < 1$ if $\rho \geq 0.86$.

ρ	0.2	0.5	0.8	0.86	0.9	1
R_0	-3.214	-1.74	-0.266	0.028	0.225	0.716

Now we take the parameters of the system as d=0.29, α=3.1, β=4.7, b=3.1, γ=1.5. k=6.5, μ=0.19, Then the endemic equilibrium exists if ρ exceeds 0.24. The behavior of endemic equilibrium for the different values of ρ is tabulated as below:

ρ	S^*	I^*	R^*	x^*	y^*	R_0
0.24	10.17	0.112	0.107	0.44	0.42	1.60
0.3	6.54	1.13	0.82	5.30	3.88	10.29
0.4	4.77	2.14	1.09	10.53	5.35	24.76
0.5	4.09	2.98	1.12	14.25	5.33	39.24
0.6	3.73	3.78	1.08	17.16	4.89	53.72
0.7	3.51	4.55	1.00	19.54	4.32	68.19
0.8	3.35	5.31	0.92	21.53	3.73	82.67
0.9	3.24	6.06	0.82	23.24	3.16	97.14
1	3.1598	6.8073	0.72256	24.72	2.62	111.62

6. CONCLUSION

One of the main issues in the study of behavior of epidemics is the analysis of steady states of the model and their stability. If the trivial or zero equilibrium is globally asymptotically stable, then, the disease does not persist, whatever the initial number of infective in the population.

An SIRS model with an asymptotically homogeneous transmission function was analyzed by Pathak et.al. [11]. They assumed that the entire interaction population becomes infectious. However, it is a little bit away from reality. We have modified this assumption by taking a part of this population as infected by introducing a new control parameter and we have carried out the global qualitative analysis of an SIR epidemic model. By numerical simulation, it has been observed that increasing value of ρ increase I^* where as S^* and R^* decrease. If we substitute the control parameter as unity, our model is in full agreement with that of Pathak et.al.[11].

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