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Roughness Based on the Closed Sub Hyper Group in a Join Space

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ABSTRACT
In this paper, we study and investigate the upper and the lower approximation with respect to a closed subhypergroup in a join space and generalize some similar results which are proved in a hypergroup.

Keywords: *Hypergroup; Join space; Rough set.*

1. INTRODUCTION AND PRELIMINARY NOTES

The concept of n-group was introduced by Drnte [1928] some 80 years ago and it proved to be a successful topic for investigations, with relevant applications in various fields e.g. physics, quantum group theory, codes, topology, etc. Another field which proved to be relevant was that of algebraic hyperstructures introduced by Marty with applications in geometry, graphs and hypergraphs, binary relations, lattices, fuzzy and rough sets, automata, cryptography, codes, artificial intelligence, probabilities etc. (see [1934]). Recently, Davvaz and Vougiouklis [2006] have established a connection between the two domains in the form of an extension of the concept of n-ary groups to the concept of n-ary hypergroups, which has also proved to be of great interest. Some applications of the n-ary hypergroups to lattices and to binary relations are analyzed by Leoreanu-Fotea and Davvaz in [2008]. One other theory that this paper relates to is rough sets theory, introduced by Pawlak [1982] which represents a mathematical tool for dealing with vagueness or uncertainty. An algebraic approach of rough set was proposed by Bonikowaski [1995], Iwinski [2010]. On the other hand, the approximation theory was studied within the context of the various algebraic structures as follows. Biswas and Nanda [1994] introduced the rough subgroup notion. Generalized rough sets based on relations were studied by Zhu in [2007]. Finally, we should mention the connections between rough sets and algebraic hyperstructures. In [2000], Corsini introduced and analyzed a class of algebraic hypergrupoizi associated to rough sets.

The present paper is a generalized study of the approximation theory in a special hypergroups which is called join space with respect to a closed subhypergroup, as done by Leoreanu-Fotea (see [2010]).

Now, we would like to introduce some basic notions and results about hypergroup (see [2007, 2008, 2010]) which are required in the sequel of our work and hence presented in brief.

Let H be a non-empty set and $o: H\times H \to P^*(H)$ be a hyperoperation. The couple (H,o) is called a hypergroupoid.

For any two non-empty subset A and B of H and $x\in H$, we define

$$AoB = \bigcup_{a\in A,\, b\in B} aob,\ Aox = Ao\{x\} \quad and\ xoB = \{x\}oB.$$

a. A hypergroupoid (H,o) is called a semihypergroup if for all a,b,c of H we have $(aob)oc = ao(boc)$, which means that

$$\bigcup_{u\in aob} uoc = \bigcup_{v\in boc} aov.$$

b. A hypergroupoid (H,o) is called a quasihypergroup if for all $a\in H$ we have $aoH = Hoa = H$.

The above condition is also called the reproduction axiom.

c. A hypergroupoid (H,o) which is both a semihypergroup and a quasihypergroup is called a hypergroup.

Moreover, if $xoy = yox$ for all $x,y\in H$, we said that (H,o) is a commutative hypergroup.

A hypergroup for which the hyperproduct of any two elements has exactly one element is a group.

A non-empty subset K of a hypergroup (H,o) is called a subhypergroup if for all a of K we have $aoK = Koa = K$ and it is called normal subhypergroup if $xoK = Kox$ for all $x\in H$.

Definition 1.1[2008]

Let (H,o) be a hypergroup and (K,o) be a subhypergroup of it, we say that K is:

(i) Closed on the left(on the right) if for all k_1, k_2 of K and $x \in H$, from $k_1 \in xok_2 (k_1 \in k_2 ox$, respectively), it follows that $x \in K.$

(ii) Invertible on the left(on the right) if for all x, y of H, from $x \in Koy(x \in yoK)$, it follows that $y \in Kox$ ($y \in xoK$, respectively).

(iii) Ultraclosed on the left(on the right) if for all $x \in H$, we have
$Kox \bigcap (H \setminus K)ox = \phi$
$(xoK \bigcap xo(H \setminus K) = \phi).$

(iv) Conjugable on the right if it is closed on the right and for all $x \in H$, there exists $x^{'} \in H$ such that $x^{'}ox \subseteq K.$

We say K is closed (invertible, ultraclosed, conjugable) if it is closed (invertible, ultraclosed, conjugable respectively) on the left and on the right.

Theorem 1.2. [2008]

Let (H,o) be a hypergroup and K be a subhypergroup of H. Then

- K is invertible on the right if and if $\{xoK\}_{x\in H}$ is a partition of $H.$
- If subhypergroup K is conjugable, then it is ultraclosed.
- If a subhypergroup K is ultraclosed, then it is invertible.
- If a subhypergroup K is invertible, then it is closed.

Example 1.3. [2007]

Let (A,o) be a hypergroup, $H = A \bigcup T$, where T is a set with at least three elements and $A \bigcup T = \phi.$ We define the hyperopration $\otimes$ on H, as follows:

if $(x, y) \in A^2$, then $x \otimes y = xoy$;
if $(x,t) \in A \times T$, then $x \otimes t = t \otimes x = t$;
if $(t_1, t_2) \in T \times T$, then $t_1 \otimes t_2 = t_2 \otimes t_1 = A \bigcup (T - \{t_1, t_2\}).$

Then $(H,\otimes)$ is a hypergroup and $(A,\otimes)$ is a ultraclosed, non-conjugable subhypergroup of H .

There are several useful examples about hypergroups in [2007].

In order to define a join space, we need the following notation:

If a,b are elements of a hypergroup (H,o), then we denote

$$a/b = \{x \in H \mid a \in xob\}.$$

Moreover, by A/B we intend the set $\bigcup_{a\in A, b\in B} a/b.$

Definition 1.4:

A commutative hypergroup (H,o) is called a join space if the following implication holds for all elements a,b,c,d of H:

$$a/b \bigcap c/d \neq \phi \Rightarrow aod \bigcap cob \neq \phi.$$

A join space (H,o) is called geometric if then exists $x \in H$ such that $xox = \{x\} = x/x.$

Example 1.5. [2007]

(1) Let $(L,\wedge,\vee)$ be a distributive lattice. If for all a,b of L we define

$$aob = \{x \in L \mid x = (a \wedge b) \vee (a \wedge x) \vee (b \wedge x)\},$$

then (L,o) is a non geometrical join space in which every element is an identity.

(2) Let V be a vector space over an ordered field F . If for all a,b of V we define

$$aob = \{\lambda a + \mu b \mid \lambda > 0, \mu > 0, \lambda + \mu = 1\},$$

then (V,o) is a geometrical join space , called an affine join space over $F.$

Definition 1.6:

If N is a closed subhypergroup of a join space H and $\{x, y\} \subseteq H$, then we define the following binary relation:

$$x J_N y \Leftrightarrow xoN \bigcap yoN \neq \phi.$$

Theorem 1.7. [2007]

J_N is an equivalence relation on H and the equivalence class of an element a is $J_N(a) = \frac{aoN}{N}.$ In particular $J_N(a) = N$ for all $a \in N.$

An element $e \in H$ is called an scaler identity if $xoe = eox = x$ for all $x \in H$.

Theorem 1.8. [2007]

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If (H,o) is a join space and N is a closed subhypergroup of H, then the quotient $(H/J_N,\otimes)$ is join space with a scalar identity, where for all $\overline{a},\overline{b}$ of H/J_N, we have

$$\overline{a}\otimes\overline{b}=\{\overline{c}\mid c\in aob\}.$$

2. LOWER AND UPPER APPROXIMATION IN A JOIN SPACE

We begin this section, by recalling the notion of a rough set [1982].

Let H be a non-empty set and $A\subseteq H$. If θ is an equivalence relation on H, then the pair $(\underline{\theta}(A),\overline{\theta}(A))$ is called the rough set of A, with respect to θ, where

$$\underline{\theta}(A)=\left\{x\in H\mid [x]_\theta\subseteq A\right\},\ \overline{\theta}(A)=\left\{x\in H\mid [x]_\theta\bigcap A\neq\phi\right\}.$$

$\underline{\theta}(A)$ is called the lower approximation of A, while $\overline{\theta}(A)$ is called the upper approximation of A.

Notice that we denote the equivalence class of $x\in H$ by $[x]_\theta$.

Let us consider now (H,o) a join space, $A\subseteq H$, and N a closed subhypergroup. By Theorem 1.7, $\theta=J_N$ is an equivalence relation on H. We denote,

$$L_N(A)=\{x\in H\mid J_N(x)\subseteq A\},$$
$$U_N(A)=\{x\in H\mid J_N(x)\bigcap A\neq\phi\}.$$

$L_N(A)$ is called the lower approximation of A with respect to N, while $U_N(A)$ is called the upper approximation of A with respect to N. The pair $(L_N(A),U_N(A))$ is called the rough set of A with respect to N. If we consider the equivalence relation J_N, defined in the above paragraph, then for all $x\in H$, we have $[x]_{J_N}=\dfrac{xoN}{N}=J_N(x)$, $L_N(A)=\underline{J}_N(A)$ and $U_N(A)=\overline{J}_N(A)$.

According to the above definition and remark, we obtain the following basic properties.

Theorem 2.1:

Let N be a closed subhypergroup of a join space (H,o), $A,B\subseteq H$ and J_N is the equivalence relation defined in the above paragraph, then we have,

(1) $L_N(A)\subseteq A\subseteq U_N(A)$;
(2) $L_N(\phi)=\phi=U_N(\phi)$;
(3) $L_N(H)=H=U_N(H)$;
(4) if $A\subseteq B$, then $L_N(A)\subseteq L_N(B)$ and $U_N(A)\subseteq U_N(B)$;
(5) $L_N(L_N(A))=L_N(A)$;
(6) $U_N(U_N(A))=U_N(A)$;
(7) $U_N(L_N(A))=L_N(A)$;
(8) $L_N(U_N(A))=U_N(A)$;
(9) $U_N(H-A)=H-L_N(A)$;
(10) $L_N(A\bigcap B)=L_N(A)\bigcap L_N(B)$;
(11) $L_N(H-A)=H-U_N(A)$;
(12) $U_N(A\bigcup B)=U_N(A)\bigcup U_N(B)$;
(13) $U_N(A\bigcap B)\subseteq U_N(A)\bigcap U_N(B)$;
(14) $L_N(A\bigcup B)\supseteq L_N(A)\bigcup L_N(B)$;
(15) $U_N(J_N(x))=L_N(J_N(x))$ for all $x\in H$.

Definition 2.2:

A subset A of a join space (H,o) is called definable with respect to N if $U_N(A)=A=L_N(A)$.

Remark 2.3:

Let A be subset of a join space (H,o) and $x\in H$. Let N be a closed subhypergroup.
Then $L_N(A)$, $U_N(A)$, $J_N(x)$ are definable sets with respect to N.

Remark 2.4:

Let N_1,N_2 be closed subhypergroup of a join space (H,o). If $N_1\subseteq N_2$, then

(i) $U_{N_2}(N_1)=N_2$.
(ii) $L_{N_2}(A)\subseteq L_{N_1}(A)$.
(iii) $U_{N_1}(A)\subseteq U_{N_2}(A)$.

Theorem 2.5:

Let N be a closed subhypergroup of a join space (H,o) and $A,B\in P^*(H)$, where $P^*(H)$ is the set of all nonempty subsets of H, then

$$U_N(A)oU_N(B)=U_N(AoB).$$

Proof:

"$\subseteq$". Let $x \in U_N(A)oU_N(B)$, then there are $x_1 \in U_N(A)$ and $x_2 \in U_N(B)$, such that $x \in x_1ox_2$. It follows that there are $u_1 \in J_N(x_1) \bigcap A$ and $u_2 \in J_N(x_2) \bigcap B$. Therefore $J_N(x_1) = J_N(u_1)$ and $J_N(x_2) = J_N(u_2)$. By Theorem 158(i) page 70 in [1993], $x \in x_1ox_2 \subseteq J_N(u_1)oJ_N(u_2) \subseteq J_N(u_1ou_2)$.

It follows that there exists $u \in u_1ou_2$, such that $x \in J_N(u)$, which means $J_N(x) = J_N(u)$. We have $u \in u_1ou_2 \subseteq AB$ and $u \in J_N(x)$, hence $x \in U_N(AoB)$.

"$\supseteq$". Let $x \in U_N(AoB)$, then there are $u_1 \in A$, $u_2 \in B$, $u \in u_1ou_2$, such that $x \in J_N(u) \subseteq J_N(u_1)oJ_N(u_2)$. Hence there are $x_1 \in J_N(u_1)$ and $x_2 \in J_N(u_2)$, such that $x \in x_1ox_2$. It follows that $u_1 \in A \bigcap J_N(x_1)$ and $u_2 \in B \bigcap J_N(x_2)$, which means $x_1 \in U_N(A)$ and $x_2 \in U_N(B)$. Therefore $x \in x_1ox_2 \subseteq U_N(A)oU_N(B)$.

Theorem 2.6:

Let N be a closed subhypergroup of a join space (H, o) and $A, B \in P^*(H)$, where $P^*(H)$ is the set of all nonempty subsets of H, then

$$L_N(A)oL_N(B) \subseteq L_N(AoB).$$

Proof:

Let $x \in L_N(A)oL_N(B)$, then there are $a \in L_N(A)$ and $b \in L_N(B)$, such that $x \in aob$. It follows that there are $J_N(a) \subseteq A$ and $J_N(b) \subseteq B$. Therefore $x \in aob \subseteq J_N(a)oJ_N(B) \subseteq AoB$.
It results that $L_N(A)oL_N(B) \subseteq L_N(AoB)$.

The following example shows that $L_N(A)oL_N(B) = L_N(AoB)$ is not true in general.

Example 2.7:

According to Example 1.3, suppose $A = Z_2$ and the hyperopration $o : Z_2 \times Z_2 \rightarrow P(Z_2)$ where $xoy = \{x, y\}$ for all $x, y \in Z_2$ and $T = \{a, b, c\}$ and $A \bigcap T = \phi$. Then $(H, \otimes)$ is a join space and $N = A$ is a closed subhypergroup which makes the following partition:

$$E_1 = Z_2, E_2 = \{a\}, E_3 = \{b\}, E_4 = \{c\}.$$

Suppose $M_1 = \{0, a\}, M_2 = \{1, a\}$. We can obtain $L_N(M_1oM_2) = H$, $L_N(M_1) = \{a\}$, $L_N(M_2) = \{b\}$ and $L_N(M_1)oL_N(M_2) = \{0, 1, b, c\}$. So, $L_N(M_1)oL_N(M_2) \neq L_N(M_1oM_2)$.

We obtain the following corollaries by using of the above theorems which have been proved in [2010][Theorem 3.5, 3.6].

Corollary 2.8:

Let N be a normal invertible subhypergroup of a join space (H, o) and $A, B \in P^*(H)$, where $P^*(H)$ is the set of all nonempty subsets of H, then $U_N(A)oU_N(B) = U_N(AoB)$.

Corollary 2.9:

Let N be a a normal invertible subhypergroup of a join space (H, o) and $A, B \in P^*(H)$, where $P^*(H)$ is the set of all nonempty subsets of H, then

$$L_N(A)oL_N(B) \subseteq L_N(AoB).$$

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