Vol 3, No.12 Dec 2013 ISSN 2225-7217
ARPN Journal of Science and Technology

On an Optimal Control Problem For a Class of Distributed Parameter Systems

[1,2]M. H. Farag, [2]A. B. Shamardan, [2]H. H. Saleh
[1]Mathematics Department, Faculty of Science, Taif University
Hawia(888) Taif, Saudi Arabia
[2]Math. Dept, Faculty of Science, Minia University, Egypt

Abstract—
This paper treats the problem of optimal control of a distributed parameter system governed by parabolic equation with control in the coefficents of the equation. The existence and uniqueness theorem for the solution of the considered problem is studied. The sufficient differentiability conditions of the functional and its gradient formulae based on solving the adjoint system is investigated. The outlined of the algorithm for solving the optimal control problem is given. Numerical results for constrained optimal control parabolic problems are presented in figures.

Keywords: Optimal control, Parabolic Equations, Distributed parameter systems, Gradient formulae, Penalty function method.

I. INTRODUCTION

Physical problems are often described by differential models either linear or nonlinear. There is an abundance of transformations of various types that appear in the literature of engineering, physics and mathematics are generaly aimed at obtaining some sort of simplifiction of differential models [1-2]. A wealth of literature exists on theoritical and computational aspects of control problems for partial differential equations [3-7].

In this paper we treat the problem of optimal control of a distributed parameter system governed by parabolic equation with control in the coefficents of the equation. The existence and uniqueness theorem for the solution of the considered problem is studied. The sufficient differentiability conditions of the functional and its gradient formulae based on solving the adjoint system is investigated. The outline of the algorithm for solving the optimal control problem is given. Numerical results for constrained optimal control parabolic problems are presented in figures.

II. STATEMENT OF THE CONTROL PROBLEM

Let D is a bounded domain of the n-dimensional Euclidean space E_n; Γ be the boundary of D, assumed to be sufficiently smooth; ν is the exterior unit normal of Γ; $T > 0$ is a given number; $x = (x_1, \cdots, x_n)$ is an arbitrary point of D;$\Omega = D \times (0, T]$; $S = \Gamma \times (0, T]$. The function spaces $v_2^{1,0}(\Omega), W_2^{1,0}(\Omega)$ and $W_2^{1,1}(\Omega)$ are described in [8].

Let's a problem on minimization of the functional

$$J_\alpha(v) = \|u - f_0\|^2_{L_2(S)} + \alpha\|v - \omega\|^2_{L_2^m(\Omega)} \quad (1)$$

on the set $V = \{v : v = (v_0(x,t), v_1(x,t), v_2(x,t)); v_i \in L_\infty(\Omega), v_i(x,t) \in M_i, i = 0,1; v_2 \in U_2 \subset L_2^{m_2}(\Omega)\}$ and $Y = L_\infty^{(m_0)}(\Omega) \times L_\infty^{(m_1)}(\Omega) \times L_\infty^{(m_2)}(\Omega)$ where M_0, M_1 and U_2 are bounded closed sets of E_{m_0}, E_{m_1} and $L_2^{m_2}(\Omega)$ respectively, under the condition

$$\frac{\partial u}{\partial t} - \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}[\lambda_{ij}(x,t,v_0)\frac{\partial u}{\partial x_j}] + \sum_{i=1}^{n} B_i(x,t,v_1)\frac{\partial u}{\partial x_i} = f(x,t,v_2), (x,t) \in \Omega, \quad (2)$$

$$u(x,0) = \phi(x), x \in D, \quad (3)$$

$$\sum_{i,j=1}^{n} \lambda_{ij}(x,t,v_0)\frac{\partial u}{\partial x_j}\cos(\nu, x_i)|_S = g(\zeta,t), (x,t) \in S \quad (4)$$

where $\alpha \geq 0$ is given,$\phi \in L_2(D), g(\zeta,t) \in L_2(S)$, $\omega = \omega(x,t) = (\omega_0(x,t), \omega_1(x,t), \omega_2(x,t))$ and $\omega_i \in L_2^{m_2}(\Omega), i = 0,1,2$ are given functions.

Now,we give some assumptions as follows:

Assumption 1: $L_p^{(m)}(\Omega)$ is the space of m-dimentional vector-valued functions $v(x,t) = (v^1(x,t), \cdots, v^m(x,t))$ with the norm $\|v\|^2_{L_p^{(m)}(\Omega)} = \sum_{i=1}^m \|v^i\|^2_{L_p^{(m)}(\Omega)}, p \geq 1$.

Assumption 2: The functions $B_i(x,t,v_1), \lambda_{ij}(x,t,v_0), i = \overline{1,n}, f(x,t,v_2)$ satisfy a Lipschitz condition for v_0, v_1, v_2.

Assumption 3: The first derivatives of the functions $B_i(x,t,v_1)$, $\lambda_{ij}(x,t,v_0), i,j = \overline{1,n}$ and $f(x,t,v_2)$ with respect to v satisfy the Carathedory conditions in their domain of definition.

Assumption 4: The operators $\frac{\partial B_i(x,t,v_1)}{\partial v_1}, i = \overline{1,n}, \frac{\partial \lambda_{ij}(x,t,v_0)}{\partial v_0}, i,j = \overline{1,n}, \frac{\partial f(x,t,v_2)}{\partial v_2}$ are bounded on $L_\infty^{(m_k)}(\Omega), k = 0,1,2$.

Definition 0: The problem of finding the function $u = u(x,t;v) \in V_2^{0,1}(\Omega)$ from condition (2)-(4) at given $v(x,t) \in V$ is called the reduced problem.

Definition 1: A function $u = u(x,t) \in V_2^{1,0}(\Omega)$ is said to be a solution of the problem (2)-(4), if for all $\eta = \eta(x,t) \in W_2^{1,1}(\Omega)$ the equation

$$\int_\Omega [-u\frac{\partial \eta}{\partial t} + \sum_{i,j=1}^n \lambda_{ij}(x,t,v_0)\frac{\partial u}{\partial x_i}\frac{\partial \eta}{\partial x_i} - \sum_{i=1}^n B_i(u,v_1)(\frac{\partial u}{\partial x_i})\eta(x,t) - f(x,t,v_2)\eta(x,t)]dxdt$$

$$= \int_D \phi(x)\eta(x,0)dx + \int_S g(\zeta,t)\eta(\zeta,t)d\zeta dt, \tag{5}$$

is valid and $\eta(x,T) = 0$.

It is proved in [8] that, under the foregoing assumptions, a reduced problem (2)-(4) has a unique solution and

$$\|u\|^2_{V_2^{1,0}(\Omega)} + \|u\|^2_{L_2(S)} \le C_1[\|\phi\|^2_{L_2(D)} + \|g\|^2_{L_2(S)} + \|f\|^2_{L_2(\Omega)}]$$

where C_1 is a certain constant.

III. Well-posedness and variation of the functional

Allowing for this remark the proof methods of the papers [9-11] we proved the following statement:

Theorem 1: Under the above assumptions for every solution of the reduced problem (2)-(4) the following estimate is valid:

$$\|\Delta u\|_{V_2^{1,0}(\Omega)} \le C_2[\|\sqrt{\sum_{i=1}^n (\sum_{j=1}^n \Delta\lambda_{ij}\frac{\partial u}{\partial x_j})^2}\|_{L_2(\Omega)} + \|\Delta f - \sum_{i=1}^n \Delta B_i \frac{\partial u}{\partial x_i}\|_{L_2(\Omega)}], \tag{6}$$

where $\Delta u(x,t) = u(x,t;v+\delta v) - u(x,t;v), \delta u(x,t) \in W_2^{1,1}(\Omega)$, $\Delta\lambda_{ij} = \lambda_{ij}(x,t,v_0+\delta v_0) - \lambda_{ij}(x,t,v_0)$,$\Delta B_i = B_i(x,t,v_1+\delta v_1) - B_i(x,t,v_1), i,j=\overline{1,n}$,$\Delta f = f(x,t,v_2+\delta v_2) - f(x,t,v_2)$ and $C_2 \ge 0$ is a constant.

Corollary 1: Under the above assumptions, the right part of estimate (6) converges to zero at $\|\Delta v\|_Y \to 0$, therefore $\|\Delta u\|_{V_2^{1,0}(\Omega)} \to 0$ when $\|\delta v\|_Y \to 0$. Hence from the theorem on trace [12] we get $\|\Delta u\|_{L_2(S)} \to 0$ when $\|\delta v\|_Y \to 0$.

Theorem 2: The functional $J_0(v) = \int_S [u(\zeta,t) - f_0(\zeta,t)]^2 d\zeta dt$ is continuous on V.

Theorem 3: For any $\alpha \ge 0$ the problem (2)-(4) has at least one solution.

Theorem 4: There exists a dense set K of $L_2^m(\Omega)$ such that for any $\omega \in K$ the problem (1)-(4) for $\alpha > 0$ has a unique solution.

Let's view the following adjoint problem an definition of function $\Theta = \Theta(x,t)$ from the conditions

$$\frac{\partial \Theta}{\partial t} + \sum_{i,j=1}^n \frac{\partial}{\partial x_j}[\lambda_{ij}(x,t,v_0)\frac{\partial \Theta}{\partial x_i}] + \sum_{i=1}^n \frac{\partial B_i(x,t,v_1)}{\partial x_i}\Theta(x,t) = 0, (x,t) \in \Omega, \tag{7}$$

$$\Theta(x,T) = 0, \quad x \in D, \tag{8}$$

$$\sum_{i,j=1}^n \lambda_{ij}(x,t,v_0)\frac{\partial \Theta}{\partial x_i}\cos(\nu,x_i)|_S = -2[u(\zeta,t) - f_0(\zeta,t)], (\zeta,t) \in S, \tag{9}$$

where $u = u(x,t)$ is the solution of reduced problem corresponding to $v \in V$.

On the basis of the above assumptions and the results [13] follows that for every $v \in V$ the solution of problem (7)-(8) is existed, unique and $|\frac{\partial \Theta}{\partial x_i}| \le C_3, i = \overline{1,n}$ almost at all $(x,t) \in \Omega, \forall v \in V$, where C_3 is a certain constant.

Definition 2: For each $v \in V$, a function $\Theta(x,t;v)$ is a solution of the adjoint problem (7)-(9) belonging to the control v iff

(i) $\Theta(x,t;v) \in V_2^{1,0}(\Omega)$,

(ii) The integral identity

$$\int_\Omega \Theta(x,t)\frac{\partial \gamma(x,t)}{\partial t}dxdt + 2\int_S [u(\zeta,t) - f_0(\zeta,t)]\gamma(x,t)d\zeta dt$$

$$= -\int_\Omega \sum_{i,j=1}^n \lambda_{ij}(x,t,v_0)\frac{\partial \Theta}{\partial x_i}\frac{\partial \gamma(x,t)}{\partial x_j}dxdt - \int_\Omega \sum_{i=1}^n \frac{\partial B_i(x,t,v_1)}{\partial x_i}\Theta(x,t)\gamma(x,t)dxdt, \tag{10}$$

is valid for all $\gamma_1 \in W_2^{1,1}(\Omega)$ with $\gamma(x,0) = 0$.

For every $v \in V$, define $H(u,\Theta,v)$ as follows:

$$H(u,\Theta,v) \equiv -[\sum_{i,j=1}^{n} \lambda_{ij}(x,t,v_0)\frac{\partial u}{\partial x_j}\frac{\partial \Theta}{\partial x_i} + \sum_{i=1}^{n} B_i(x,t,v_1)\frac{\partial u}{\partial x_i}\Theta(x,t)$$

$$-f(x,t,v_2)\Theta(x,t) + \alpha\sum_{k=0}^{2}\sum_{i=1}^{m_k}(v_k^i - \omega_k^i)^2] \quad (11)$$

where $H(u,\Theta,v)$ is called the Hamilton function of the optimal control problem (1)-(6).

Theorem 5: Let the above assumptions be satisfied. Then for any function $v = (v_0, v_1, v_2) \in V$ it is valid the following expression the first variation of the functional $J_\alpha(v)$:

$$\Delta J_\alpha(v) = \int_\Omega \langle \sum_{i,j=1}^{n} \frac{\partial \lambda_{ij}(x,t,v_0(x,t))}{\partial v_0}\frac{\partial u}{\partial x_j}\frac{\partial \Theta}{\partial x_i} + \sum_{i=1}^{n}\frac{\partial B_i(x,t,v_1(x,t))}{\partial v_1}\frac{\partial u}{\partial x_i}\Theta(x,t)$$

$$+\frac{\partial f(x,t,v_2(x,t))}{\partial v}\Theta(x,t) + 2\alpha[v(x,t) - \omega(x,t)], \delta v\rangle_{E_m} dxdt. \quad (12)$$

IV. Special Case and The Numerical Results

A. Constrained Control Problem and Modified functional

We applied the theoritical results in the above sections on a constrained parabolic optimal control problem described by distributed parameter systems and the control v is a variable on t only.

Let us consider the following parabolic distributed parameter system as follows:

$$\frac{\partial y}{\partial t} = \frac{\partial}{\partial x}(\lambda(x)\frac{\partial y}{\partial x}) + F(x,t,v(t)), (x,t) \in \Omega, \quad (13)$$

$$y(x,0) = \phi(x), \qquad 0 \le x \le l \quad (14)$$

$$\frac{\partial y}{\partial x}|_{x=0} = 0, \qquad 0 < t \le T, \quad (15)$$

$$-\lambda(x)\frac{\partial y}{\partial x}|_{x=l} = \eta[G(y(1,t)) - v(t)], \qquad 0 < t \le T, \quad (16)$$

$$\beta_0 \le y(x,t) \le \beta_1, \quad (17)$$

$$J_\alpha(v) = \int_0^l [y(x,T;v) - f_0(x)]^2 dx + \alpha\int_0^T [v(t) - \omega(t)]^2 dt \to min \quad (18)$$

The constrained optimal control problem (13)-(18) is converted to an unconstrained control problem by adding a penalty function [14] to the cost functional (18), yielding the modified functional $\Psi_{\alpha,k}(v, r_k)$

$$\Psi_{\alpha,k}(v,r_k) \equiv \Psi(v) = J_\alpha(v) + P_k(v), \quad (19)$$

where

$$\Phi^1(u) = [\max\{\beta_1 - u(x,t;v); 0\}]^2, \Phi^2(u) = [\max\{u(x,t;v) - \beta_2; 0\}]^2$$

$$P_k(v) = r_k\int_0^l\int_0^T [\Phi^1(u) + \Phi^2(u)]dxdt$$

and $r_k > 0$, k=1,2,... are positive numbers, $\lim_{k\to\infty} r_k = +\infty$.

B. Numerical Algorithm

The outlined of the algorithm for solving the unconstrained optimal control problem problem are as follows:

1- Given $It = 0, \epsilon' > 0, r_{It} > 0, \epsilon > 0$ and $v^{(It)} \in V$.

2- At each iteration It, do

  Solve state system (13)-(16), then find $u(., v^{(It)})$.

  Minimze $\Psi(v^{It})$ to find optimal control $v_*^{(It+1)}$ using PMPQI technique [15].

End do.

3- If $\|\Psi(v^{It+1}) - \Psi(v^{It})\| < \epsilon$, then Stop, else, go to Step 4.

4- Set $v^{(It+1)} = v^{(It)}, r_{It+1} = \epsilon' r_{It}, It = It + 1$ and go to Step 2.

C. Numerical Example

We consider the case of a linear profile, that is $f_0(x) = 1-x$. The graphs of exact and approximate optimal control at the third and ninth (It=3,It=9) steps of iterations are plotted in Figs. 1 and 2. One can see from Fig. 2 that at the ninth iteration step the approximate optimal control actually coincidcs with the exact control, which illustratcs the efficiency of the computation algorithm on the whole.

In figure 3,the variation of the temperature profiles in the minimization process is illustrated. The values of the approximate gradient function $\frac{\partial \Psi_\alpha(v)}{\partial v}$ by the formula $\frac{\partial \Psi_\alpha(v)}{\partial v} = \frac{\Psi_\alpha(v+h)-\Psi_\alpha(v-h)}{2h} + O(h^2)$ various number of iterations (It) are shown in Figure 4.

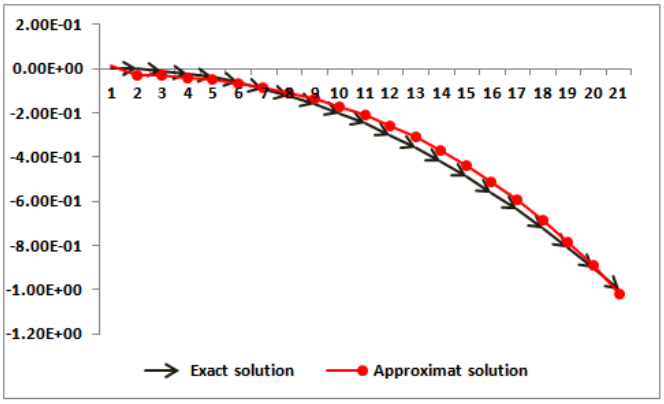

Fig. 1. Exact and approximate of optimal control function $v(t)$ at It=3

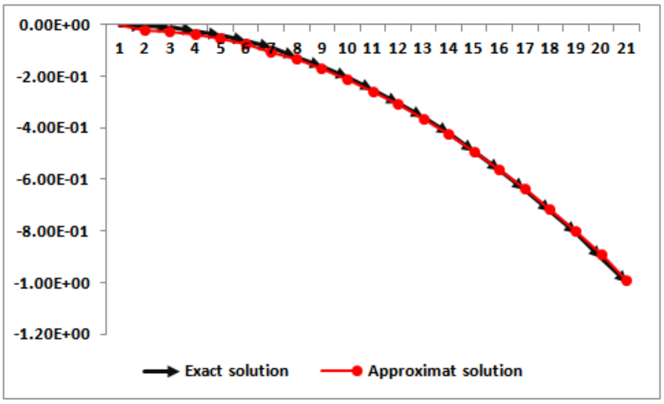

Fig. 2. Exact and approximate of optimal control function $v(t)$ at It=9

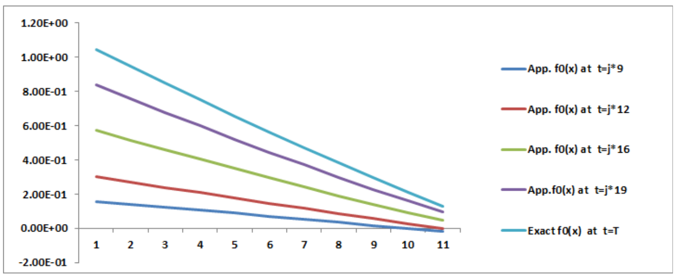

Fig. 3. The temperature profiles in the minimization process $f_0(x)$

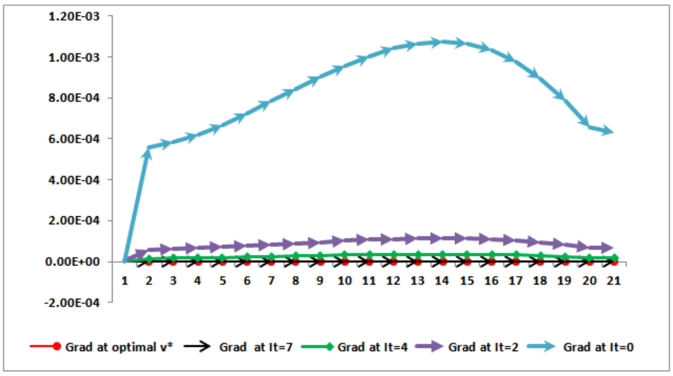

Fig. 4. Approximate gradient values various iterations

REFERENCES

[1] Li Chun-fa, Xue Yang and En-min Feng,Optimal control problem governed by semilinear parabolic equation and its algorithm,Acta Mathematicae Sinica 186 (2008) 29-40.
[2] T. B. Gardashov, Solution of inverse problems for the quasilinear heat conduction equation for the multidimensional case, J. Engrg. Phys. 61 (3) (1991) 472-478.
[3] A. D. Iskenderov and R. K. Tagiev, Optimization problems with controls in coefficients of parabolic equations,Differentsialnye Uravneniya 19 (8) (1983) 1324-1334.
[4] H. F. Guliyev and K. SH. Jabbarova, On an optimal control problem for weakly nonlinear hyperbolic equations, Acta Math. Hungar, Baku, Azerbaijan,2011.
[5] A. Hasan, B. Foss, S. I. Sagatun, B. P. Tjostheim, A. Svandal, C. Hatland, Modeling, simulation, and optimal control of oil production under gas coning conditions, SPE Journal of petroleum Technology 63 (7) (2011) 79-80.
[6] J. L. Lions, Optimal control of systems governed by partial differential equations, Springer-Verlag, Berlin-Heidelberg,1971.
[7] I. V. Bushuev, On a class of optimal control problems for parabolic equations, Sib. Math. J. 35 (1994) 887-892.
[8] O. A. Ladyzhenskaya, Boundary value problems of mathematical physics, Nauka, Moscow, 1973.
[9] M. H. Farag,On the derivation of gradient formulae by solving the conjugate boundary value problem for an optimal control problem, J. Egypt. Math. Soc. 4 (1996) 41-47.
[10] M. H. Farag,T. A. Talaat and E. K. Kamal, Existence and uniqueness solution of a class of quasilinear parabolic boundary control problems, Cubo Math. J. 15 (2) (2013) 111-119.
[11] M. H. Farag,Computing optimal control with a quasilinear parabolic partial differential equation, Surveys in Mathematics and its Applications 23 (2009) 34-44.
[12] V. P. Mikhailov, Partial differential equations,Nauka, Moscow, Russian,1983.
[13] O. A. Ladyzhenskaya, V. A. Solonnikov and N. N. Ural'tseva, Linear and quasilinear parabolic equations, Nauka, Moscow, 1973.
[14] O. Yeniay,Penalty function methods for constrained optimization with genetic algorithms, Mathematical and Computational Application (2005) 45-46.
[15] A. A. Radwan,Acceleration of partial quadratic interpolation technique, M. Sc. Thesis, University of Assuit, 1980.