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Using Square-Root Inverted Gamma Distribution as Prior to Draw Inference on the Rayleigh Distribution

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ABSTRACT

Based on a complete failure time data, the maximum likelihood and Bayesian estimator under squared error and general entropy loss functions for the scale parameter and the reliability function of the Rayleigh distribution are derived. Assessments between the estimators are investigated through a simulation study. The results indicate that, Bayes estimator under the squared error loss function performs better than the others having obtained and compared the estimators using mean squared errors and absolute biases of the estimated values.

Keywords: *Inverted Gamma, Squared Error and General Entropy Loss functions, Bayesian Inference, Simulation Study*

1. INTRODUCTION

The Rayleigh distribution is a special case of the two parameter Weibull distribution and a suitable model for life testing studies. Polovko (1968) and Dyer and Whisenand (1973) demonstrated the importance of this distribution in electro vacuum devices and communication engineering.

The cumulative distribution function (c.d.f.), the reliability function and the density function of the Rayleigh distribution are defined as

$$F(x) = 1 - \exp\left(-\frac{x^2}{2\sigma^2}\right), x \in [0, \infty), \tag{1}$$

$$R(x) = 1 - \left[1 - \exp\left(-\frac{x^2}{2\sigma^2}\right)\right] \tag{2}$$

and

$$f(x;\sigma) = \frac{x}{\sigma^2}\exp\left(-\frac{x^2}{2\sigma^2}\right), x \ge 0, \sigma > 0 \tag{3}$$

Its hazard rate is a linearly increasing function of time. Hence, when the failure times are distributed according to the Rayleigh distribution, the reliability function decreases at a much higher rate than that of the exponential reliability function does, Kim and Han (2009). In many life testing studies, it is common that the lifetimes of some test units may be recorded exactly, indicating that all the units have failed.

Inferences for the Rayleigh distribution have been discussed by several authors. Harter and Moore (1965) derived an explicit form for the MLE of σ based on type II censored data. Dyer and Whisenand (1973) considered the best linear unbiased estimator of σ based on type II censored sample. Balakrishnan (1989) showed that an approximate MLE is as efficient as the best linear unbiased estimator. Bayesian estimation and prediction problems for the Rayleigh distribution based on doublycensored sample have been considered by Fernandez (2000) and Raqab and Madi (2002). Wu et al (2006) have also considered the Bayesian estimator and prediction intervals for future observations based on progressively type II censored samples. Kim and Han (2009) considered MLE, approximate MLE and Bayes estimation procedures for the scale parameter based on a multiply type II censored sample. Guure et al (2013) considered Bayesian inference based on Weibull model for interval-censored survival data.

In this paper, our main object is to study the maximum likelihood estimation and Bayes estimation procedures for the scale parameter σ and the reliability $R(x)$ of the Rayleigh distribution based on a complete sample According to complete samples, Surles and Padgett (2001) showed that the two-parameter generalized Rayleigh distribution is quite effective in modeling strength of data and general lifetime data.

The rest of this paper is organized as follows. In Section 2, the MLE of the parameter (σ) and the reliability $R(x)$ based on a complete failure data sample are presented. In Section 3, the Bayes estimator under squared error loss and general entropy loss functions are introduced. Comparisons among the estimators are conducted through simulations in section 4, results are in section 5 and section 6 conclusion.

2. MAXIMUM LIKELIHOOD ESTIMATOR

In this section we consider the maximum likelihood estimator (MLE) of the Rayleigh distribution.

Let $x_i, \ldots, x_n$ be a random sample of size n from a Rayleigh distribution, then the likelihood function can be written as;

$$f(x_i;\sigma)=\prod_{i=1}^{n}\left[\frac{x_i}{\sigma^2}\exp\left(-\frac{x_i^{\,2}}{2\sigma^2}\right)\right] \tag{4}$$

and the log-likelihood written as

$$\ell=\sum_{i=1}^{n}x_i-n\log(\sigma^2)-\left(\frac{1}{2\sigma^2}\sum_{i=1}^{n}x_i^{\,2}\right) \tag{5}$$

The normal equation becomes;

$$\frac{d\ell}{d\sigma}=-\frac{2n}{\sigma}+\frac{\sum_{i=1}^{n}x_i}{\sigma^3}=0,$$

implying that

$$\hat{\sigma}_{MLE}=\left(\frac{1}{2n}\sum_{i=1}^{n}x_i^2\right)^{\frac{1}{2}} \tag{6}$$

Therefore maximum likelihood estimate of the reliability function from [6] is

$$\hat{R}(x,\sigma)=\left[\exp\left(-\frac{\sum_{i=1}^{n}x_i^{\,2}}{2\hat{\sigma}^2_{MLE}}\right)\right] \tag{7}$$

3. BAYESIAN ESTIMATION

Since σ, is the parameter being sort for, according to Bayesian it is a random variable, we therefore, consider the natural conjugate family of prior distributions for σ used in Fernandez (2000), as

$$\pi(\sigma)\propto\left(\frac{1}{\sigma}\right)^{2p+1}\exp\left(-\frac{\theta}{2\sigma^2}\right),\sigma>0, \tag{8}$$

where the shape parameter p > 0 and scale parameter θ > 0. This density is known as the square-root inverted gamma distribution. For θ = 0, $\pi(\sigma)$ reduces to a general class of improper priors and if p = 0 and θ = 0, then an improper prior for σ is the Jeffreys (1961) prior. Note that if $\alpha=\frac{1}{\sigma^2}$, then the density function of α has a gamma distribution with parameters p and $\frac{\theta}{2}$.

Combining equations (4) and (8), the posterior density function of σ can be obtained as

$$\pi^*(\sigma|x)=\frac{\pi(\sigma)\prod_{i=1}^{n}\left[\frac{x_i}{\sigma^2}\exp\left(-\frac{x_i^{\,2}}{2\sigma^2}\right)\right]}{\int\pi(\sigma)\prod_{i=1}^{n}\left[\frac{x_i}{\sigma^2}\exp\left(-\frac{x_i^{\,2}}{2\sigma^2}\right)\right]d\sigma} \tag{9}$$

We can therefore obtain the posterior density function under the squared error loss function from above with respect to the parameter σ. The squared error loss function is simply the posterior mean.
Hence, we have

$$\pi^*(\sigma|x)=\frac{\int u(\sigma)\pi(\sigma)\prod_{i=1}^{n}\left[\frac{x_i}{\sigma^2}\exp\left(-\frac{x_i^{\,2}}{2\sigma^2}\right)\right]d\sigma}{\int\pi(\sigma)\prod_{i=1}^{n}\left[\frac{x_i}{\sigma^2}\exp\left(-\frac{x_i^{\,2}}{2\sigma^2}\right)\right]d\sigma} \tag{10}$$

where $u(\sigma)$ represents the loss function.

When we consider the Bayes estimate of the reliability function under this loss function the posterior density will be

$$R(x)=\frac{\int u\left[\exp\left(-\frac{\sum_{i=1}^{n}x_i^{\,2}}{2\sigma^2}\right)\right]\pi(\sigma)\prod_{i=1}^{n}\left[\frac{x_i}{\sigma^2}\exp\left(-\frac{x_i^{\,2}}{2\sigma^2}\right)\right]d\sigma}{\int\pi(\sigma)\prod_{i=1}^{n}\left[\frac{x_i}{\sigma^2}\exp\left(-\frac{x_i^{\,2}}{2\sigma^2}\right)\right]d\sigma} \tag{11}$$

The general entropy loss function is asymmetric in nature, in that, it is used to determine the degree of overestimation and underestimation of a function of interest. It is a generalization of the entropy loss function. The Bayes estimator of σ, denoted by $\hat{\sigma}_{BG}$ is given as

$\hat{\sigma}_{BG}=\left[E_\sigma(\sigma)^{-k}\right]^{-\frac{1}{k}}$, provided $E_\sigma(.)$ exist and is finite.

Therefore, the posterior density functions of the parameter and the reliability function are given respectively as

$$\pi^*(\sigma|x)=\frac{\int u[\sigma]^{-k}\,\pi(\sigma)\prod_{i=1}^{n}\left[\frac{x_i}{\sigma^2}\exp\left(-\frac{x_i^{\,2}}{2\sigma^2}\right)\right]d\sigma}{\int\pi(\sigma)\prod_{i=1}^{n}\left[\frac{x_i}{\sigma^2}\exp\left(-\frac{x_i^{\,2}}{2\sigma^2}\right)\right]d\sigma}$$

(12)

and

$$R(x)_{BG} = \frac{\int u \left[\exp\left(-\frac{\sum_{i=1}^{n} x_i^2}{2\sigma^2} \right) \right]^{-k} \pi(\sigma) \prod_{i=1}^{n} \left[\frac{x_i}{\sigma^2} \exp\left(-\frac{x_i^2}{2\sigma^2} \right) \right] d\sigma}{\int \pi(\sigma) \prod_{i=1}^{n} \left[\frac{x_i}{\sigma^2} \exp\left(-\frac{x_i^2}{2\sigma^2} \right) \right] d\sigma}$$

(13)

The above equations cannot be obtained explicitly hence we apply a numerical approach proposed by Lindley (1980) to approximate the ratio of two integrals such as (6). This has been used by several authors to obtain the approximate Bayes estimators. For details see Lindley (1980) and Press (2001). Based on Lindley's approximation, the approximate Bayes estimators of σ and $\exp\left(-\frac{x^2}{2\sigma^2}\right)$ under the squared errors loss function is given according to Guure et al (2013a)

$$\hat{\sigma} = u + \frac{1}{2}\left[u_{11}\delta_{11}\right] + u_1\rho_1\delta_{11} + \frac{1}{2}\left[\ell_{30}u_1\delta_{11}^2\right]$$

Hence

$$\ell_{20} = \frac{2n}{\sigma^2} - \frac{3x^2}{\sigma^4},\ u = \sigma, u_1 = 1, u_{11} = 0$$

and

$$\delta_{11} = \left(-\ell_{20}\right)^{-1}$$

$$\ell_{30} = -\frac{4n}{\sigma^3} + \frac{12x^2}{\sigma^5}$$

and the prior is

$$\rho_1 = \frac{-\frac{\left(\frac{1}{\sigma}\right)^{2p+1}(2p+1)\exp\left(-\frac{\theta}{2\sigma^2}\right)}{\sigma}}{\left(\frac{1}{\sigma}\right)^{2p+1}\exp\left(-\frac{\theta}{2\sigma^2}\right)} + \frac{\frac{\left(\frac{1}{\sigma}\right)^{2p+1}\theta\exp\left(-\frac{\theta}{2\sigma^2}\right)}{\sigma^3}}{\left(\frac{1}{\sigma}\right)^{2p+1}\exp\left(-\frac{\theta}{2\sigma^2}\right)}$$

For the reliability function

$$u = \exp\left(-\frac{x^2}{2\sigma^2}\right), u_1 = \frac{x^2\exp\left(-\frac{x^2}{2\sigma^2}\right)}{\sigma^3},$$

$$u_{11} = -\frac{3x^2\exp\left(-\frac{x^2}{2\sigma^2}\right)}{\sigma^4} + \frac{x^4\exp\left(-\frac{x^2}{2\sigma^2}\right)}{\sigma^6}$$

Considering the general entropy loss function, we have for the parameter

$$u = (\sigma)^{-k}, u_1 = -k(\sigma)^{-k}, u_{11} = k^2(\sigma)^{-k}$$

and for the reliability function

$$u = \left[\exp\left(-\frac{x^2}{2\sigma^2}\right)\right]^{-k}, u_1 = -\frac{kx^2\left[\exp\left(-\frac{x^2}{2\sigma^2}\right)\right]^{-k}}{\sigma^3}$$

and

$$u_{11} = \frac{3kx^2\left[\exp\left(-\frac{x^2}{2\sigma^2}\right)\right]^{-k}}{\sigma^4} + \frac{k^2x^4\left[\exp\left(-\frac{x^2}{2\sigma^2}\right)\right]^{-k}}{\sigma^6}$$

4. SIMULATION STUDY

A simulation study was carried out to determine the best estimator for the scale parameter and the reliability function of the Rayleigh distribution with complete failure data. We report the results for σ = 0.5, 1.0 and 1.5 and that of n = 25, 50 and 100. In this section our main aim is to compare the Bayes estimator with the classical maximum likelihood estimator. To make the comparison more meaningful, we assume the non-informative prior on the Rayleigh parameter by taking $\theta = p = 0$ of the square-root inverted gamma distribution. This makes it an improper prior but the posterior distribution is proper. We compare the MLEs with the Bayes estimates in terms of biases and mean squared errors (MSE) for different sample sizes and for different parameter values.

All the computations are performed using the R programming language which is freely available online. We have considered for generality the general entropy loss function parameter to be $k = \pm 0.8$. Note that the generation of $R(\sigma)$ is very simple. If U follows a uniform distribution in [0, 1], then $x = \sigma\sqrt{-2\log(U)}$ follows $R(\sigma)$. Therefore, if one has a good uniform random number generator, then the generation of Rayleigh random deviate is immediate.

For each sample size we compute the MLEs of the scale parameter and the reliability function and also the Bayes estimates using Lindley's approximation. We replicate the process 1000 times and obtain the mean squared error and the absolute bias of the estimates. The results are reported in Tables 1 and 2.

5. RESULTS

Table 1: Mean Squared Error for $\hat{\sigma}$ and $\hat{R}(x)$ of the Estimators

n	σ	$\hat{\sigma}_{mle}$	$\hat{\sigma}_{bs}$	$\hat{\sigma}_{ge}$ k=0.8	$\hat{\sigma}_{ge}$ k=-0.8	$\hat{R}(x)_{mle}$	$\hat{R}(x)_{bs}$	$\hat{R}(x)_{ge}$ k=0.8	$\hat{R}(x)_{ge}$ k=-0.8
25	0.5	0.00653	0.00631	0.00633	0.00643	0.08687	0.08550	0.08641	0.08708
	1.0	0.00361	0.00320	0.00354	0.00338	0.03494	0.03392	0.04778	0.04050
	1.5	0.05800	0.02733	0.03203	0.03232	0.19081	0.17813	0.23973	0.23800
50	0.5	0.00319	0.00316	0.00315	0.00316	0.08534	0.08504	0.08480	0.08559
	1.0	0.00180	0.00173	0.00178	0.00173	0.03489	0.03371	0.04113	0.03702
	1.5	0.01870	0.01661	0.01756	0.01734	0.18912	0.16851	0.21972	0.21966
100	0.5	0.00158	0.00157	0.00157	0.00157	0.08483	0.08466	0.08448	0.08465
	1.0	0.00089	0.00087	0.00088	0.00087	0.03451	0.03125	0.03746	0.03565
	1.5	0.00926	0.00876	0.00898	0.00984	0.18806	0.13691	0.20584	0.19934

Table 2: Absolute Bias for $\hat{\sigma}$ and $\hat{R}(x)$ of the Estimators

n	σ	$\hat{\sigma}_{mle}$	$\hat{\sigma}_{bs}$	$\hat{\sigma}_{ge}$ k=0.8	$\hat{\sigma}_{ge}$ k=-0.8	$\hat{R}(x)_{mle}$	$\hat{R}(x)_{bs}$	$\hat{R}(x)_{ge}$ k=0.8	$\hat{R}(x)_{ge}$ k=-0.8
25	0.5	0.01649	0.01621	0.01624	0.01636	0.27188	0.26970	0.27136	0.27227
	1.0	0.01218	0.01140	0.01202	0.01172	0.17275	0.16853	0.20238	0.18635
	1.5	0.03978	0.03341	0.03632	0.03647	0.39589	0.37726	0.44075	0.43856
50	0.5	0.00807	0.00804	0.00802	0.00803	0.26686	0.26631	0.26603	0.26755
	1.0	0.00604	0.00591	0.00600	0.00591	0.17153	0.16843	0.18601	0.17652
	1.5	0.01953	0.01839	0.01892	0.01880	0.38976	0.36877	0.41810	0.41842
100	0.5	0.00399	0.00398	0.00398	0.00398	0.26494	0.26443	0.26430	0.26456
	1.0	0.00299	0.00297	0.00297	0.00297	0.17004	0.16228	0.17721	0.17280
	1.5	0.00967	0.00941	0.00953	0.00950	0.38701	0.33435	0.40369	0.39827

mle =maximum likelihood, bs =Bayes squared error and ge =Bayes general entropy loss

6. CONCLUSION

We obtained the Bayesian estimation approach using square-root inverted gamma prior from which we had a non-informative prior for the scale parameter of the Rayleigh distribution which was employed using squared error and general entropy loss functions via Lindley approximation. Comparisons are made between the estimators based on simulation study with mean squared errors and absolute biases.

Table 1, shows the mean squared error values of the scale parameter and the reliability function. It's been observed that, the Bayes non-informative prior estimator has the smallest mean squared error values under the squared error loss function than the others to a very large extend. As the sample size increases Bayes estimator under the general loss functions performs better than MLE but they all have their MSE values decreasing correspondingly.

The Absolute Bias of the estimated values are presented in Table 2. We observe that all the estimators also have their absolute bias values decreasing as the sample size increases but again Bayes estimator with respect to squared error loss gives very minimal bias than the others.

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