VOL. 3, NO. 10, October 2013 ISSN 2225-7217
ARPN Journal of Science and Technology

http://www.ejournalofscience.org

Generalized Polynomials Onan Unbounded Interval

[1] **Anwar Habib,** [2] **Adel S. Bahakeem**
[1, 2] Jubail Industrial College, KSA

ABSTRACT

Chlodovsky (1937) has proved some results for Bernstein Polynomials

$$B_n(x) = B_n^f(x;b_n)=\sum_{k=0}^{n} f\left(\frac{b_n k}{n}\right)\binom{n}{k}\left(\frac{x}{b_n}\right)^k\left(1-\frac{x}{b_n}\right)^{n-k}$$ on an unbounded interval.

The small modification of Bernstein Polynomials due to Kantorovic (1930) makes it possible to define a polynomial on L_{1}-norm as

$$U_n(x) = U_n^f(x,\alpha;b_n) = (n+1)\sum_{k=0}^{n}\left\{\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(tb)dt\right\} q_{n,k}\left(\frac{x}{b_n};\alpha\right)$$

where $$q_{n,k}(x;\alpha)=\binom{n}{k}\frac{x(x+k\alpha)^{k-1}(1-x)(1-x+(n-k)\alpha)^{n-k-1}}{(1+n\alpha)^{n-1}}$$ on an unbounded interval .

The object of this paper is to extend these results for lebegue integrable function in L_1 norm by a newly defined Generalized Polynomial

Keywords: *Bernstein Polynomials, Lebsgue Integrable function, L_1 norm, Generating function, Generalized Polynomials*

1. INTRODUCTION & RESULTS

If f(x) is a function defined on [0, 1] , the Bernstein polynomial $B_n^f(x)$ of f(x) is $B_n^f(x)=\sum_{k=0}^{n} f\left(\frac{k}{n}\right)p_{n,k}(x)$,

Where $p_{n,k}(x) = \binom{n}{k}(x)^k(1-x)^{n-k}$

If the function f(x) defined in the interval (0,b),b>0 .The Bernstein polynomial $B_n^f(x;b)$ for this interval is given by

$$B_n(x)=B_n^f(x;b)=\sum_{k=0}^{n} f\left(\frac{b}{n}\right)\binom{n}{k}\left(\frac{x}{b}\right)^k\left(1-\frac{x}{b}\right)^{n-k} \quad (1.1)$$

By Abel's formula (Jensen [3])

$$(x+y)(x+y+n\alpha)^{n-1} = \sum_{k=0}^{n}\binom{n}{k} x(x+k\alpha)^{k-1}y(y+n-k\alpha)n-k-1 \ldots \quad (1.2)$$

If we put $y = 1-x$, we obtain (Cheney and Sharma [2])

$$1 = \sum_{k=0}^{n}\binom{n}{k}\frac{x(x+k\alpha)^{k-1}(1-x)(1-x+(n-k)\alpha)^{n-k-1}}{(1+n\alpha)^{n-1}} \ldots \quad (1.3)$$

Thus defining

$$q_{n,k}(x;\alpha) = \binom{n}{k}\frac{x(x+k\alpha)^{k-1}(1-x)(1-x+(n-k)\alpha)^{n-k-1}}{(1+n\alpha)^{n-1}} \ldots \quad (1.4)$$

we have

$$\sum_{k=0}^{n} q_{n,k}(x;\alpha) = 1 \ldots \quad (1.5)$$

Further a small modification of the Bernstein polynomial due to Kantorovich [4] and Anwar & Saleh [5] makes it possible to approximate Lebesgue integrable function in L_1 norm by a newly defined polynomial

$$U_n^\alpha(f,x) = (n+1)\sum_{k=0}^{n}\left\{\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)dt\right\} q_{n,k}(x;\alpha) \quad (1.6)$$

where

$q_{n,k}(x;\alpha)$ is as (1.4)

Let the function f(x) be defined on the interval (0,b),b>0. To obtain a modified polynomial $U_n^f(x,\alpha;b)$ for this interval, we make the substitution

$y= xb^{-1}$ in the polynomial $U_n^\Phi(y)$ of the function $\Phi(y) = f(by)$, $0\le y\le 1$

and obtain in this way

VOL. 3, NO. 10, October 2013 ISSN 2225-7217
ARPN Journal of Science and Technology

http://www.ejournalofscience.org

$$U_n(x) = U_n^f(x,\alpha;b) = (n+1)\sum_{k=0}^{n}\{\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(tb)dt\}\, q_{n,k}\left(\frac{x}{b};\alpha\right) \text{----} \quad (1.7)$$

Where

$$q_{n,k}\left(\frac{x}{b};\alpha\right)=\binom{n}{k}\frac{\left(\frac{x}{b}\right)\left(\frac{x}{b}+k\alpha\right)^{k}\left(1-\frac{x}{b}\right)\left(1-\frac{x}{b}+(n-k)\alpha\right)^{n-k-1}}{(1+n\alpha)^{n-1}} \text{-------------------------} \quad (1.8)$$

Chlodovsky (1937) has proved his results by assuming $b=b_n$ is a function of n, which increases to $+\infty$ with n and f(x) defined in the infinite interval $0 \leq x < \infty$.

Theorem 1.1:

If $b_n=0(n)$ and the function f(x) is bounded in $[0, +\infty)$, say $|f(x)| \leq M$, then $B_n(x) \to f(x)$ holds at any point of continuity of the function f(x) .

Theorem 1.2:

If $b_n=0(n)$

and

$$M(b_n)e^{-\alpha n/bn} \to 0,$$

for each $\alpha > 0$, then $B_n(x) \to f(x)$ holds at each point of continuity of the function f(x).

In this paper our object is to improve the above results by taking the new polynomial $U_n(x)$ for lebesgue integrable function in L_1 norm which may be stated as follows

Theorem 1.3:

If $b_n=0(n)$ and the function f(x) is bounded lebesgue integrable in $[0, \infty)$, say $|f(x)| \leq M$, then $U_n(x) \to f(x)$ holds at any point of continuity of the function f(x) .

Theorem 1.4:

If $b_n=0(n)$

and

$$M(b_n)e^{-\beta n/b_n} \to 0, \quad (1.9)$$

for each $\beta > 0$,then $U_n(x) \to f(x)$ holds at each point of continuity of the function $f(x)$.

2. LEMMAS

In order to proof our result we need the following Lemmas

Lemma 2.1:

[6]**:** For all values of x

$$\sum_{k=0}^{n} k\, q_{n,k}(x;\alpha) \leq \frac{!+n\alpha}{1+\alpha} nx - \frac{n(n-1)x\alpha}{1+2\alpha}$$

Lemma 2.2:

[6]**:** For all values of x

$$\sum_{k=0}^{n} k(k-1)q_{n,k}(x;\alpha) \leq n(n-1)[(x+2\alpha)\{\frac{1+n\alpha}{(1+2\alpha)^2} - \frac{(n-2)\alpha}{(1+3\alpha)^2}\} + (n-2)\alpha^2\{\frac{1+n\alpha}{(1+3\alpha)^3} - \frac{(n-3)\alpha}{(1+4\alpha)^3}\}]$$

Lemma2.3:

[6]: For all values of $x \epsilon [0,1]$ and for $\alpha=\alpha_n=0(\frac{1}{n})$

We have $$(n+1)\sum_{k=0}^{n}\{\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}}(t-x)^2 dt\}\, q_{n,k}(x;\alpha) \leq \frac{x(1-x)}{n}.$$

Lemma 2.4:

If $0 \leq x \leq 1$, the inequality,

$$0 \leq z \leq \frac{3}{2}\left(\frac{x(1-x)}{n}\right)^{\frac{1}{2}} \text{-------------------------} \quad (2.1)$$

Implies

$$(n+1)\sum_{|t-x| \geq 2z\left(\frac{x(1-x)}{n}\right)^{\frac{1}{2}}} \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} q_{n,k}(x)dt \leq 2e^{-z^2}. \text{------------} \quad (2.2)$$

Proof of lemma 2.4:

Let Φ be the generating function of the polynomial

$$T= \sum_{k=0}^{n}(k-nx)q_{n,k}(x:\alpha),$$

which may be defined as

$$\Phi=\Phi_n(u,s) = \sum_{s=0}^{\infty}\frac{1}{s!}T_{ns}(x)u^s$$

$$=\sum_{k=0}^{n} p_{n,k}(x;\alpha)\sum_{s=0}^{\infty}\frac{1}{s!}(k-nx)^s u^s$$

$$=\sum_{k=0}^{n} e^{u(k-nx)}\binom{n}{k}\frac{x(x+k\alpha)^{k-1}(1-x)(1-x+(n-k)\alpha)^{n-k-1}}{(1+n\alpha)^{n-1}}$$

$$=\frac{e^{-nxu}}{(1+n\alpha)^{n-1}}[(1-x)(1-x+n\alpha)^{n-1} + nx1-x)(1-x+(n-1)\alpha)^{n-2}e^{u}$$

$$+ \frac{n(n-1)}{2!}x(x+2\alpha)(1-x)(1-x+(n-2)\alpha)^{n-3}e^{2u} + \cdots \dots \dots \dots + (1-x)\,x(x+n\alpha)^{n-1}e^{nu}$$

VOL. 3, NO. 10, October 2013 ISSN 2225-7217
ARPN Journal of Science and Technology

http://www.ejournalofscience.org

$\Phi = e^{-nxu}(1 - x + xe^{u})^{n}$, for α=α_n=0($\frac{1}{n}$)

and therefore

$\Phi = [e^{-xu}(1 - x + xe^{u})]^{n}$----------------------(2.3)

To prove our result we first show that for$|u| \le \frac{3}{2}$, the inequality

$\Phi \le exp\{nx(1-x)u^2\}$…………………….(2.4)

holds.

For, (2.3) can be written as

$$\Phi = [xe^{u(1-x)} + (1-x)e^{-ux})]^{n}$$

But since

$xe^{u(1-x)} + (1-x)e^{-ux} = \sum_{\nu=0}^{\infty} \frac{u^{\nu}}{\nu!}[x(1-x) + (1-x)(-x)\nu]$

$\le 1 + \sum_{\nu=2}^{\infty} \frac{u^{\nu}}{\nu!}[x(1-x) + (1-x)(-x)^{\nu}]$

$\le 1 + x(1-x)\sum_{\nu=2}^{\infty} \frac{|u|^{\nu}}{\nu!}$

$\le 1 + x(1-x)\frac{u^2}{2}(1 + \frac{|u|}{3} + \frac{|u^2|}{3^2} + \ldots\ldots\ldots\ldots)$

$= 1 + x(1-x)\frac{u^2}{2}(1 - \frac{1}{3}|u|)^{-1}$

$\le 1 + x(1-x)u^2$ for $|u| \le \frac{3}{2}$

$\le e^{x(1-x)u^2}$ as $e^{k} > k+1$

and hence

$\Phi \le [e^{x(1-x)u^2}]^{n}$

$= \exp\{n\, x(1-x)u^2\}$ which is (2.4) .

Therefore if

$\psi = \psi_n(u,x) = \sum_{k=0}^{n} e^{u|k-nx|}\, q(x;\alpha)$ ------------------ (2.5)

then we obtain for 0 ≤ u ≤ $\frac{3}{2}$

$\psi \le \psi_n(u,x) + \Psi_n(-u,x)$ and therefore , for α=α_n=0($\frac{1}{n}$) , we have

$\psi \le \exp 2\{n\, x(1-x)u^2\}$. ----------------- (2.6)

now we get our required result, we note that for c≥0 and u≥0

$$(n+1)\sum_{exp[u|k-nx|]\ge c\psi} \left(\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} dt\right) q_{n,k}(x;\alpha)$$

$\le \frac{1}{c\psi}(n+1)\sum_{k=0}^{n} \{\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)dt\} e^{u|k-nx|}\, q_{n,k}(x;\alpha) \le \frac{1}{c}$

Now if we put c=$\frac{1}{2}z^2$, we obtain

$$(n+1)\sum_{exp[u|k-nx|]\ge\frac{1}{2}e^{z^2}\psi} \left(\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} dt\right) q_{n,k}(x;\alpha) \le 2e^{-z^2}$$

or

$$(n+1)\sum_{|k-nx|\ge z^2u^{-1}+nx(1-x)u} \left(\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} dt\right) q_{n,k}(x;\alpha) \le 2e^{-z^2}$$

Since for the given range of t$|k-nx| \sim |t-x|$, we have

$$(n+1)\sum_{|t-x|\ge z^2u^{-1}n^{-1}+nx(1-x)u} \left(\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} dt\right) q_{n,k}(x;\alpha) \le 2e^{-z^2} \quad \text{-------} \quad (2.7)$$

Since 2.1 can be written as

$0 \le z[nx(1-x)]^{-\frac{1}{2}} \le \frac{3}{2}$

But (2.7)holds for 0 ≤ u ≤ $\frac{3}{2}$ and therefore for u= $z[nx(1-x)]^{-\frac{1}{2}}$, we have

$$(n+1)\sum_{|t-x|\ge z\left[\frac{x(1-x)}{n}\right]^{\frac{1}{2}}+z\left[\frac{x(1-x)}{n}\right]^{\frac{1}{2}}} \left(\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} dt\right) q_{n,k}(x;\alpha) \le 2e^{-z^2}$$

$$(n+1)\sum_{|t-x|\ge 2z\left[\frac{x(1-x)}{n}\right]^{\frac{1}{2}}} \left(\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} dt\right) q_{n,k}(x;\alpha) \le 2e^{-z^2}$$

this completes the proof of lemma.

3. PROOF OF THEOREMS

Proof of theorem 1.3:

We have

$|U_n(x) - f(x)| \le (n+1)\sum_{k=0}^{n} \{\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} |f(tb) - f(x)|dt\}\, q_{n,k}(\frac{x}{b};\alpha)$

Let ϵ>0 be arbitrary and choose infinitesimally small δ>0 such that

$|f(x) - f(x')| < \epsilon$ for $|x - x'| < \delta$

then

$$|U_n(x) - f(x)| \le (n+1)\sum_{|bnt-x|<\delta} \{\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} |f(tb) - f(x)|dt\}\, q_{n,k}\left(\frac{x}{b};\alpha\right)$$

$$+(n+1)\sum_{|bnt-x|\geq\delta}\{\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}}|f(tb)-f(x)|dt\}\, q_{n,k}\left(\frac{x}{b};\alpha\right)$$

$=I_1+I_2$ --------------------------------------- (3.1)

$I_1=(n+$
$1bnt-x<\delta\{kn+1k+1n+1ftb-fxdt\}qn,kxb;\alpha$

$$<\epsilon \quad (n+1)\sum_{|bnt-x|<\delta}\{\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}}dt\}\, q_{n,k}\left(\frac{x}{b};\alpha\right)$$

$=\epsilon$ ------------------------------------ (3.2)

To calculate I_2, we put $u=\frac{x}{b_n}$ and then we have

$I_2=(n+$
$1bnt-x\geq\delta\{kn+1k+1n+1ftb-fxdt\}qn,kxb;\alpha$

$$\leq 2M\,(n+1)\sum_{|t-u|\geq\frac{\delta}{bn}}\{\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}}dt\}\, q_{n,k}(u;\alpha)$$

$$\leq 2M\left(\frac{\delta}{bn}\right)^{-2}(n+1)\sum_{k=0}^{n}\{\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}}(t-u)^2dt\}\, q_{n,k}(u;\alpha)$$

$\leq 2M\left(\frac{\delta}{bn}\right)^{-2}\frac{u(1-u)}{n}$ for all $\alpha=0(\frac{1}{n})$ by Lemma (2.1),

$\leq 2M\,\frac{\frac{x}{bn}}{n(\frac{\delta}{bn})^2}$ for all large n & $\alpha=0(\frac{1}{n})$ since $b_n=o(n)$,

$<\epsilon$ -- (3.3)

Hence
$|U_n(x)-f(x)|\leq\epsilon+\epsilon=2\epsilon$
which completes the proof of theorem 1.3.

Proof of theorem 1.4:

Proceeding as in theorem 1.3 we obtain
$|U_n(x)-f(x)|\leq\epsilon+2M(b_n)$

$$(n+1)\sum_{|t-u|\geq\frac{\delta}{b_n}}\{\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}}dt\}\, q_{n,k}(u;\alpha)$$

The second term can be easily estimated by means of lemma (2.2), if

$z=\delta_n(2b_n)^{-1}\left(\frac{u(1-u)}{n}\right)^{-\frac{1}{2}}$,

the condition (2.1) satisfied if we assume, for instance, $\delta<2$ xand that n is sufficiently large .

Hence by (1.6) we obtain
$|U_n(x)-f(x)|\leq\epsilon+2M(b_n)\exp(-z^2)$
$=\epsilon+2M(b_n)\exp\{-\delta^2.n[4\frac{x}{bn}\left(1-\frac{x}{bn}\right)]^{-1}\}$
$=\epsilon+\epsilon=2\epsilon$ for large n
which completes the proof of the theorem 1.4.

4. CONCLUSION

In this paper we have improved the results of Chlodovsky by taking the new Generalized Polynomials $U_n(x)$ instead of Bernstein Polynomials $B_n(x)$.

REFERENCES

[1] Chlodovsky, I (1937). Sur le development des fonctions definiesdansun interval infini en series de polynomes de M S Bernstein composition math, 4, 380-93.

[2] Cheney, E.W. , and Sharma, A.(1964).” On a generalization of Bernstein polynomials”.Rev. Mat. Univ. Parma(2),**5**,77-84.

[3] Jensen, J. L. W. A. (1902). “Sur uneidentité Abel et surd’autressformulesamalogues”.

[4] Kantorovic, L A (1930). Sur certainsdeveloppementssuivaint les polynomes de la forme de S Bernstein I , II, C R Acad. Sci. USSR , 20,563-68,595- 600 .

[5] Anwar Habib and S Umar (1980) “On Generalized Bernstein Polynomials” Indian J. pure appl. Math. , 11(2) , 177-189 .

[6] Anwar Habib and Saleh Shehri (2012) “ on Generalized Polynomials I “International Journal of Engineering Research and Development e-ISSN : 2278 -067 X , p – ISSN : 2278 – 800 X vol. 5 Issue 4 (2012) pp. 18-26.