

Generalized Polynomials On an Unbounded Interval

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ABSTRACT

Chlodovsky (1937) has proved some results for Bernstein Polynomials

$$B_n(x) = B_n^f(x; b_n) = \sum_{k=0}^n f\left(\frac{kb_n}{n}\right) \binom{n}{k} \left(\frac{x}{b_n}\right)^k \left(1 - \frac{x}{b_n}\right)^{n-k} \text{ on an unbounded interval.}$$

The small modification of Bernstein Polynomials due to Kantorovic (1930) makes it possible to define a polynomial on L_1 -norm as

$$U_n(x) = U_n^f(x, \alpha; b_n) = (n+1) \sum_{k=0}^n \left\{ \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t) dt \right\} q_{n,k} \left(\frac{x}{b_n}; \alpha\right)$$

where

$$q_{n,k}(x; \alpha) = \binom{n}{k} \frac{x(x+k\alpha)^{k-1} (1-x)(1-x+(n-k)\alpha)^{n-k-1}}{(1+n\alpha)^{n-1}} \text{ on an unbounded interval.}$$

The object of this paper is to extend these results for Lebesgue integrable function in L_1 norm by a newly defined Generalized Polynomial

Keywords: Bernstein Polynomials, Lebesgue Integrable function, L_1 norm, Generating function, Generalized Polynomials

1. INTRODUCTION & RESULTS

If $f(x)$ is a function defined on $[0, 1]$, the Bernstein polynomial $B_n^f(x)$ of $f(x)$ is $B_n^f(x) = \sum_{k=0}^n f\left(\frac{k}{n}\right) p_{n,k}(x)$,

Where $p_{n,k}(x) = \binom{n}{k} (x)^k (1-x)^{n-k}$

If the function $f(x)$ defined in the interval $(0, b), b > 0$. The Bernstein polynomial $B_n^f(x; b)$ for this interval is given by

$$B_n(x) = B_n^f(x; b) = \sum_{k=0}^n f\left(\frac{kb}{n}\right) \binom{n}{k} \left(\frac{x}{b}\right)^k \left(1 - \frac{x}{b}\right)^{n-k} \dots \dots \dots (1.1)$$

By Abel's formula (Jensen [3])

$$(x+y)(x+y+n\alpha)^{n-1} = \sum_{k=0}^n \binom{n}{k} x(x+k\alpha)^{k-1} y(y+n-k\alpha)^{n-k-1} \dots (1.2)$$

If we put $y = 1-x$, we obtain (Cheney and Sharma [2])

$$1 = \sum_{k=0}^n \binom{n}{k} \frac{x(x+k\alpha)^{k-1} (1-x)(1-x+(n-k)\alpha)^{n-k-1}}{(1+n\alpha)^{n-1}} \dots \dots (1.3)$$

Thus defining

$$q_{n,k}(x; \alpha) = \binom{n}{k} \frac{x(x+k\alpha)^{k-1} (1-x)(1-x+(n-k)\alpha)^{n-k-1}}{(1+n\alpha)^{n-1}} \dots \dots (1.4)$$

we have

$$\sum_{k=0}^n q_{n,k}(x; \alpha) = 1 \dots (1.5)$$

Further a small modification of the Bernstein polynomial due to Kantorovich [4] and Anwar & Saleh [5] makes it possible to approximate Lebesgue integrable function in L_1 norm by a newly defined polynomial

$$U_n^\alpha(f, x) = (n+1) \sum_{k=0}^n \left\{ \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t) dt \right\} q_{n,k}(x; \alpha) \dots \dots \dots (1.6)$$

where

$$q_{n,k}(x; \alpha) \text{ is as (1.4)}$$

Let the function $f(x)$ be defined on the interval $(0, b), b > 0$. To obtain a modified polynomial $U_n^f(x, \alpha; b)$ for this interval, we make the substitution

$$y = xb^{-1} \text{ in the polynomial } U_n^\Phi(y) \text{ of the function } \Phi(y) = f(by), 0 \leq y \leq 1$$

and obtain in this way

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$$U_n(x) = U_n^f(x, \alpha; b) = (n+1) \sum_{k=0}^n \left\{ \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(tb) dt \right\} q_{n,k} \left(\frac{x}{b}; \alpha \right) \quad (1.7)$$

Where

$$q_{n,k} \left(\frac{x}{b}; \alpha \right) = \binom{n}{k} \frac{\left(\frac{x}{b}\right)^k \left(\frac{x}{b} + k\alpha\right)^{n-k} (1 - \frac{x}{b} + (n-k)\alpha)^{n-k-1}}{(1+n\alpha)^{n-1}} \quad (1.8)$$

Chlodovsky (1937) has proved his results by assuming $b=b_n$ is a function of n , which increases to ∞ with n and $f(x)$ defined in the infinite interval $0 \leq x < \infty$.

Theorem 1.1:

If $b_n=0(n)$ and the function $f(x)$ is bounded in $[0, +\infty)$, say $|f(x)| \leq M$, then $B_n(x) \rightarrow f(x)$ holds at any point of continuity of the function $f(x)$.

Theorem 1.2:

If $b_n=0(n)$ and $M(b_n)e^{-\alpha n/b_n} \rightarrow 0$, for each $\alpha > 0$, then $B_n(x) \rightarrow f(x)$ holds at each point of continuity of the function $f(x)$.

In this paper our object is to improve the above results by taking the new polynomial $U_n(x)$ for lebesgue integrable function in L_1 norm which may be stated as follows

Theorem 1.3:

If $b_n=0(n)$ and the function $f(x)$ is bounded lebesgue integrable in $[0, \infty)$, say $|f(x)| \leq M$, then $U_n(x) \rightarrow f(x)$ holds at any point of continuity of the function $f(x)$.

Theorem 1.4:

If $b_n=0(n)$ and $M(b_n)e^{-\beta n/b_n} \rightarrow 0$, for each $\beta > 0$, then $U_n(x) \rightarrow f(x)$ holds at each point of continuity of the function $f(x)$.

2. LEMMAS

In order to proof our result we need the following Lemmas

Lemma 2.1:

[6]: For all values of x
 $\sum_{k=0}^n k q_{n,k}(x; \alpha) \leq \frac{1+n\alpha}{1+\alpha} nx - \frac{n(n-1)x\alpha}{1+2\alpha}$

Lemma 2.2:

[6]: For all values of x

$$\sum_{k=0}^n k(k-1)q_{n,k}(x; \alpha) \leq n(n-1)[(x+2\alpha)\left\{\frac{1+n\alpha}{(1+2\alpha)^2} - \frac{(n-2)\alpha}{(1+3\alpha)^2}\right\} + (n-2)\alpha^2\left\{\frac{1+n\alpha}{(1+3\alpha)^3} - \frac{(n-3)\alpha}{(1+4\alpha)^3}\right\}]$$

Lemma 2.3:

[6]: For all values of $x \in [0,1]$ and for $\alpha = \alpha_n = 0(\frac{1}{n})$

We have $(n+1) \sum_{k=0}^n \left\{ \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} (t-x)^2 dt \right\} q_{n,k}(x; \alpha) \leq \frac{x(1-x)}{n}$.

Lemma 2.4:

If $0 \leq x \leq 1$, the inequality,

$$0 \leq z \leq \frac{3}{2} \left(\frac{x(1-x)}{n}\right)^{\frac{1}{2}} \quad (2.1)$$

Implies

$$(n+1) \sum_{|t-x| \geq 2z \left(\frac{x(1-x)}{n}\right)^{\frac{1}{2}}} \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} q_{n,k}(x) dt \leq 2e^{-z^2} \quad (2.2)$$

Proof of lemma 2.4:

Let Φ be the generating function of the polynomial

$$T = \sum_{k=0}^n (k-nx)q_{n,k}(x; \alpha),$$

which may be defined as

$$\begin{aligned} \Phi = \Phi_n(u, s) &= \sum_{s=0}^{\infty} \frac{1}{s!} T_{ns}(x) u^s \\ &= \sum_{k=0}^n p_{n,k}(x; \alpha) \sum_{s=0}^{\infty} \frac{1}{s!} (k-nx)^s u^s \\ &= \sum_{k=0}^n e^{u(k-nx)} \binom{n}{k} \frac{x(x+k\alpha)^{k-1} (1-x)(1-x+(n-k)\alpha)^{n-k-1}}{(1+n\alpha)^{n-1}} \\ &= \frac{e^{-nxu}}{(1+n\alpha)^{n-1}} [(1-x)(1-x+n\alpha)^{n-1} + nx(1-x)(1-x+(n-1)\alpha)^{n-2} e^u \\ &\quad + \frac{n(n-1)}{2!} x(x+2\alpha)(1-x)(1-x+(n-2)\alpha)^{n-3} e^{2u} + \dots + (1-x)x(x+n\alpha)^{n-1} e^{nu} \end{aligned}$$

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$$\Phi = e^{-nxu} (1 - x + xe^u)^n, \text{ for } \alpha = \alpha_n = 0\left(\frac{1}{n}\right)$$

and therefore

$$\Phi = [e^{-xu} (1 - x + xe^u)]^n \dots \dots \dots (2.3)$$

To prove our result we first show that for $|u| \leq \frac{3}{2}$, the inequality

$$\Phi \leq \exp\{nx(1-x)u^2\} \dots \dots \dots (2.4)$$

holds.

For, (2.3) can be written as

$$\Phi = [xe^{u(1-x)} + (1-x)e^{-ux}]^n$$

But since

$$\begin{aligned} xe^{u(1-x)} + (1-x)e^{-ux} &= \sum_{v=0}^{\infty} \frac{u^v}{v!} [x(1-x) + (1-x)(-x)^v] \\ &\leq 1 + \sum_{v=2}^{\infty} \frac{u^v}{v!} [x(1-x) + (1-x)(-x)^v] \\ &\leq 1 + x(1-x) \sum_{v=2}^{\infty} \frac{|u|^v}{v!} \\ &\leq 1 + x(1-x) \frac{u^2}{2} \left(1 + \frac{|u|}{3} + \frac{|u|^2}{3^2} + \dots \dots \dots\right) \\ &= 1 + x(1-x) \frac{u^2}{2} \left(1 - \frac{1}{3}|u|\right)^{-1} \\ &\leq 1 + x(1-x)u^2 \text{ for } |u| \leq \frac{3}{2} \\ &\leq e^{x(1-x)u^2} \text{ as } e^k > k + 1 \end{aligned}$$

and hence

$$\begin{aligned} \Phi &\leq [e^{x(1-x)u^2}]^n \\ &= \exp\{nx(1-x)u^2\} \text{ which is } \dots \dots \dots (2.4). \end{aligned}$$

Therefore if

$$\Psi = \Psi_n(u, x) = \sum_{k=0}^n e^{u|k-nx|} q(x; \alpha) \dots \dots \dots (2.5)$$

then we obtain for $0 \leq u \leq \frac{3}{2}$

$$\begin{aligned} \Psi &\leq \Psi_n(u, x) + \Psi_n(-u, x) \text{ and therefore, for } \\ \alpha &= \alpha_n = 0\left(\frac{1}{n}\right), \text{ we have } \\ \Psi &\leq \exp\{2nx(1-x)u^2\} \dots \dots \dots (2.6) \end{aligned}$$

now we get our required result, we note that for $x \geq 0$ and $u \geq 0$

$$\begin{aligned} (n+1) \sum_{\exp\{|u| |k-nx| \geq c\}} \left(\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} dt\right) q_{n,k}(x; \alpha) \\ \leq \frac{1}{c} (n+1) \sum_{k=0}^n \left\{ \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t) dt \right\} e^{u|k-nx|} q_{n,k}(x; \alpha) \leq \frac{1}{c} \end{aligned}$$

Now if we put $c = \frac{1}{2} z^2$, we obtain

$$(n+1) \sum_{\exp\{|u| |k-nx| \geq \frac{1}{2} z^2\}} \left(\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} dt\right) q_{n,k}(x; \alpha) \leq 2e^{-z^2}$$

or

$$(n+1) \sum_{|k-nx| \geq z^2 u^{-1} + nx(1-x)u} \left(\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} dt\right) q_{n,k}(x; \alpha) \leq 2e^{-z^2}$$

Since for the given range of $|k-nx| \sim |t-x|$, we have

$$(n+1) \sum_{|t-x| \geq z^2 u^{-1} + nx(1-x)u} \left(\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} dt\right) q_{n,k}(x; \alpha) \leq 2e^{-z^2} \dots \dots \dots (2.7)$$

Since 2.1 can be written as

$$0 \leq z[nx(1-x)]^{\frac{1}{2}} \leq \frac{3}{2}$$

But (2.7) holds for $0 \leq u \leq \frac{3}{2}$ and therefore for $u = z[nx(1-x)]^{\frac{1}{2}}$, we have

$$(n+1) \sum_{|t-x| \geq z \left[\frac{x(1-x)}{n} \right]^{\frac{1}{2}} + z \left[\frac{x(1-x)}{n} \right]^{\frac{1}{2}}} \left(\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} dt\right) q_{n,k}(x; \alpha) \leq 2e^{-z^2}$$

$$(n+1) \sum_{|t-x| \geq 2z \left[\frac{x(1-x)}{n} \right]^{\frac{1}{2}}} \left(\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} dt\right) q_{n,k}(x; \alpha) \leq 2e^{-z^2}$$

this completes the proof of lemma.

3. PROOF OF THEOREMS

Proof of theorem 1.3:

We have

$$|U_n(x) - f(x)| \leq (n+1) \sum_{k=0}^n \left\{ \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} |f(tb) - f(x)| dt \right\} q_{n,k}(x; \alpha)$$

Let $\epsilon > 0$ be arbitrary and choose infinitesimally small $\delta > 0$ such that

$$|f(x) - f(x')| < \epsilon \text{ for } |x - x'| < \delta$$

$$\text{then } |U_n(x) - f(x)| \leq (n$$

$$\begin{aligned} + 1) \sum_{|bnt-x| < \delta} \left\{ \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} |f(tb) - f(x)| dt \right\} q_{n,k} \left(\frac{x}{b}; \alpha \right) \end{aligned}$$

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$$\begin{aligned}
 &+(n+1) \sum_{|bnt-x| \geq \delta} \left\{ \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} |f(tb) - f(x)| dt \right\} q_{n,k} \left(\frac{x}{b}; \alpha \right) \\
 &= I_1 + I_2 \quad \text{-----} \quad (3.1)
 \end{aligned}$$

$$\begin{aligned}
 I_1 &= (n+1) \sum_{|bnt-x| < \delta} \left\{ \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} |f(tb) - f(x)| dt \right\} q_{n,k} \left(\frac{x}{b}; \alpha \right)
 \end{aligned}$$

$$\begin{aligned}
 &< \epsilon (n+1) \sum_{|bnt-x| < \delta} \left\{ \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} dt \right\} q_{n,k} \left(\frac{x}{b}; \alpha \right) \\
 &= \epsilon \quad \text{-----} \quad (3.2)
 \end{aligned}$$

To calculate I_2 , we put $u = \frac{x}{b_n}$ and then we have

$$\begin{aligned}
 I_2 &= (n+1) \sum_{|bnt-x| \geq \delta} \left\{ \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} |f(tb) - f(x)| dt \right\} q_{n,k} \left(\frac{x}{b}; \alpha \right)
 \end{aligned}$$

$$\begin{aligned}
 &\leq 2M (n+1) \sum_{|t-u| \geq \frac{\delta}{b_n}} \left\{ \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} dt \right\} q_{n,k}(u; \alpha) \\
 &\leq 2M \left(\frac{\delta}{b_n} \right)^{-2} (n+1) \sum_{k=0}^n \left\{ \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} (t-u)^2 dt \right\} q_{n,k}(u; \alpha) \\
 &\leq 2M \left(\frac{\delta}{b_n} \right)^{-2} \frac{u(1-u)}{n} \text{ for all } \alpha = 0 \left(\frac{1}{n} \right) \text{ by Lemma (2.1),} \\
 &\leq 2M \frac{\frac{x}{b_n}}{n \left(\frac{\delta}{b_n} \right)^2} \text{ for all large } n \ \& \ \alpha = 0 \left(\frac{1}{n} \right) \text{ since } b_n = o(n), \\
 &< \epsilon \quad \text{-----} \quad (3.3)
 \end{aligned}$$

Hence

$$|U_n(x) - f(x)| \leq \epsilon + \epsilon = 2\epsilon$$

which completes the proof of theorem 1.3.

Proof of theorem 1.4:

Proceeding as in theorem 1.3 we obtain

$$|U_n(x) - f(x)| \leq \epsilon + 2M(b_n)$$

$$(n+1) \sum_{|t-u| \geq \frac{\delta}{b_n}} \left\{ \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} dt \right\} q_{n,k}(u; \alpha)$$

The second term can be easily estimated by means of lemma (2.2), if

$$z = \delta_n (2b_n)^{-1} \left(\frac{u(1-u)}{n} \right)^{-\frac{1}{2}},$$

the condition (2.1) satisfied if we assume, for instance, $\delta < 2$ and that n is sufficiently large.

Hence by (1.6) we obtain

$$\begin{aligned}
 |U_n(x) - f(x)| &\leq \epsilon + 2M(b_n) \exp(-z^2) \\
 &= \epsilon + 2M(b_n) \exp\left\{-\delta^2 \cdot n \left[4 \frac{x}{b_n} \left(1 - \frac{x}{b_n}\right)\right]^{-1}\right\} \\
 &= \epsilon + \epsilon = 2\epsilon \quad \text{for large } n
 \end{aligned}$$

which completes the proof of the theorem 1.4.

4. CONCLUSION

In this paper we have improved the results of Chlodovsky by taking the new Generalized Polynomials $U_n(x)$ instead of Bernstein Polynomials $B_n(x)$.

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