VOL. 2, NO. 5, June 2012 ISSN 2225-7217
ARPN Journal of Science and Technology

http://www.ejournalofscience.org

Viscous Flow through Porous Media between Two Moving Parallel Disks: Exact Solutions and Stability Analysis

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ABSTRACT

In this paper the viscous fluid flow through porous media moving between two moving parallel disks is considered. These disks are considered to be moving towards or away from each other with a constant velocity q. Here we obtain the exact analytical solution by using the method of separation of variables of the Navier-Stokes equations with the Darcy term for flow through porous medium. The graph of the stationary solutions is depicted. The motion near the stagnation point, the periodical one-dimensional perturbation is applied and found that the movement of disks determines the stability of the solution. That is when the disks are moving towards the solution is stable and disks are moving in opposite directions the solution is unstable up to certain level after which the amplitude decreases rapidly, owing to dissipation. Stationary solutions in the form of jets are studied, which gives two components the first one is the motion corresponding to the potential flow and second one is the jet behavior (non-potential flow component).

Keywords: *Viscous fluid, exact solution, porous media.*

1. INTRODUCTION

The viscous fluid flow between moving parallel porous or non-porous discs has been studied extensively in the last few decades due to immense applications in the industries, for example, the aerodynamics extrusion of plastic sheets, the cooling of a large metallic plate in a cooling bath, in the design of thrust bearings, radial diffusers etc.

The pioneering study of liquid motion between two parallel disks, moving towards each other or in opposite directions with a constant velocity was carried by Aristov and Gitaman[1]. Aristov and Gitaman gave a formulation of the problem that the motion of a viscous incompressible liquid between two parallel sides, moving towards each other or in opposite directions, is considered. The description of possible conditions of motion is based on the exact solution of the Navier-Stokes equations. The stability of the motion is analyzed for different initial perturbations.

There is a large class of processes which can be considered from the mathematical point of view as the motion of liquid between two parallel disks, moving towards each other or in opposite directions with a constant velocity. This include such processes as the motion of liquid through a hydraulic pump, digging through slurry and the motion of underground water can also be explained with a help of the this model (figure 1) It is observed for the different types of hydro dynamical problems, the mathematical descriptions are the same. So it can be explained for the water motion in a hydraulic pump (when impermeable disks are moving towards or apart each other). similarly to the motion of under ground water motion through porous media. These problems are interesting because of their analytical solutions.

Suppose two parallel disks are placed in water and start moving them towards each other or away from each other assuming the length of the disks to be much larger than the distance between them. Even with a bigger view, we can see that when the disks are approaching each other the effort required is smaller than that for separation when the disks are moving apart. This can be explained by the different types of the fluid motion; when the disks are moving towards, it is potential; when the disks are moving apart it is rotational.

This is similar to the explanation of the types of possible instability of motion. Craik et al [3] described a procedure for finding classes of exact solutions of the Revilin-Erickson equation of unsteady non-Newtonian fluids. These solutions consist of a basic flow with spatially uniform rates of strain, disturbance of a planar form and a series type. The disturbance is continuously disturbed by the basic flow but a not remains of planar form at all times, which is similar to the formulation was given by Lagnado et al [4].

In the present work, the porous media is introduced between two disks. The flow in the porous media deals with the analysis in which the differential equation governing the fluid motion is based on the Darcy's law which accounts for the drag exerted by the porous medium. The governing nonlinear differential equations are solved analytically using separation of variables method. Furthermore, an instability analysis has been performed.

VOL. 2, NO. 5, June 2012 ISSN 2225-7217
ARPN Journal of Science and Technology

http://www.ejournalofscience.org

2. FORMULATION OF THE PROBLEM

We introduce a Cartesian coordinate system of the motion of viscous incompressible fluid between two parallel disks moving towards each other in the case when $d << l$ (where d is the distance between the disks and l is the length of the disks). Let us assume that the horizontal velocity does not depend on the vertical coordinate where as the vertical velocity depends linearly on the distance between the disks.

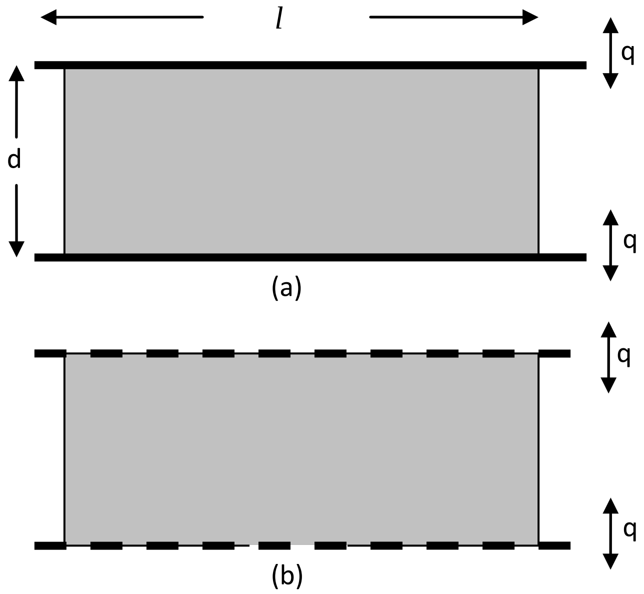

Fig 1 (a): Moving impermeable disks and (b) Moving permeable disks.

In this case the Navier-Stokes equations have the following form.

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=2q \qquad (2.1)$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\upsilon\left[\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right]-\frac{\upsilon}{\kappa}u \qquad (2.2)$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial y}+\upsilon\left[\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right]-\frac{\upsilon}{\kappa}v \qquad (2.3)$$

Where the velocity components are represented as

$$u=u(x,y,t) \quad v=v(x,y,t) \quad w=-2q\ z$$

and

$$p=\left(-4q^2+\frac{\upsilon}{\kappa}(2q)\right)\frac{z^2}{2}+p(x,y,t) \qquad (2.4)$$

Eliminating the pressure term and introducing ω the vorticity, given by $\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}=\omega$.

Therefore the vorticity equation is given by

$$\frac{\partial \omega}{\partial t}+u\frac{\partial \omega}{\partial x}+\omega\frac{\partial u}{\partial x}+v\frac{\partial \omega}{\partial y}+\omega\frac{\partial v}{\partial y}=\upsilon\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right]\omega-\frac{\upsilon}{\kappa}\omega \qquad (2.5)$$

3. METHOD OF ANALYSIS

For convenience of analysis let us select the potential component from the horizontal components of the velocity and introduce the flow function

$$u=qx+\frac{\partial \psi}{\partial y} \qquad (3.1)$$

$$v=qy-\frac{\partial \psi}{\partial x} \qquad (3.2)$$

where ψ is the stream function. Now equation (2.1) is satisfied identically and equation (2.5). The vorticity equation of motion will be in the following form:

$$\frac{\partial \omega}{\partial t}-\{\psi,\omega\}=-q\left[\frac{\partial}{\partial y}(y\omega)+\frac{\partial}{\partial x}(x\omega)\right]+\upsilon\Delta\omega-\frac{\upsilon}{\kappa}\omega \qquad (3.3)$$

where $\Delta=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$ and $\{\psi,\omega\}$ denotes the Poisson brackets:

$$\{\psi,\omega\}=\frac{\partial \psi}{\partial x}\frac{\partial \omega}{\partial y}-\frac{\partial \psi}{\partial y}\frac{\partial \omega}{\partial x}$$

One of the solutions of the equation (3.3) is $\psi=0$, which corresponds to fluid potential motion, known as the motion near the stagnation point. Following the work of Craik [2], to investigate the stability of this solution let us consider the periodical one-dimensional perturbation $\delta\psi$. This perturbation is expressed by the following equation.

$$\psi=\hat{\psi}+\delta\psi=k^{-2}(t)A(t)\cos(k(t)x) \qquad (3.4)$$

To see the change of the vorticity in this duration we put the stream function $\delta\psi$ into equation (3.4) and comparing the coefficients of $x\sin(k(t)x)$ and $\cos(k(t)x)$ on both sides we get

$$\frac{dk}{dt}=-q\,k(t) \qquad (3.5)$$

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$$\frac{dA}{dt} = -2q A(t) - \upsilon k^2(t)A(t) - \frac{\upsilon}{\kappa}A(t) \qquad (3.6)$$

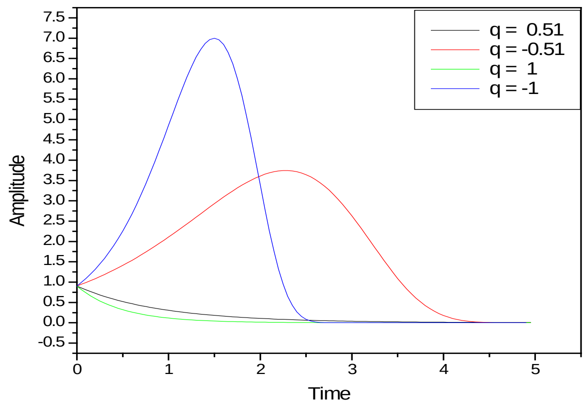
Equations (3.5), (3.6) are subject to the initial conditions $k(0) = \text{constant}$, $A(0)=\text{constant}$. The solution for $t \geq 0$ is

$$k(t) = k(0)e^{-qt} \qquad (3.7)$$

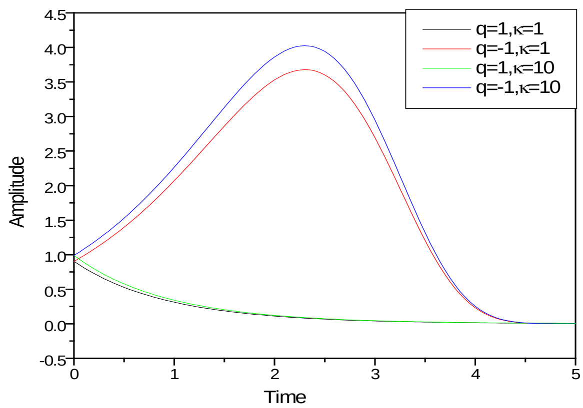
$$A(t) = A(0)\exp\left[-2qt - \frac{\upsilon t}{\kappa} + \left(-1 + e^{-2qt}\right)\upsilon\frac{k^2(0)}{2q}\right] \qquad (3.8)$$

where $k(0)$, $A(0)$ are free constants, determining the amplitude and wavelength at the initial point of time. The sign of q in equations

to dissipation.



Amplitude versus Time for different velocity of the disks



Amplitude versus Time with different κ values

Fig 2: Amplitude versus Time for different velocity of the disks and for different κ values

4. RESULTS AND DISCUSSION

4.1 Stability analysis

Let us consider the case when the flow function perturbation has the following form

$$\psi = \hat{\psi} + \delta\psi = \frac{A_1(t)}{k_1^2(t)}\cos[k_{11}(t)x + k_{12}(t)y] \qquad (4.1)$$

substituting this in equation (3.3) we get

$$k_1^2 = k_1^2(0)e^{-2qt} \qquad (4.2)$$

$$A_1(t) = A_1(0)\exp\left[-2qt - \frac{\upsilon t}{\kappa} + \left(-1 + e^{2qt}\right)\frac{\upsilon k_1^2(0)}{2q}\right] \qquad (4.3)$$

In general,

Let us consider the case when the flow function perturbation has the following form

$$\delta\psi = \sum_{i=1}^{N}\frac{A_i(t)}{k_i^2(t)}\cos(k_{i1}(t)x + k_{i2}(t)y) \qquad (4.4)$$

Provided that $k_{i1}^2 + k_{i2}^2 = k^2 \qquad \forall i$

Remark 1. If $q > 0$, the solution is stable with both the amplitude and the wave number k decreasing in the course of time. Otherwise if $q < 0$, the solution is unstable. However the amplitude increases until

$$t = \sum_{i=1}^{N}\left(-\frac{1}{2q}\right)\ln\left|\frac{q}{\upsilon k^2(0)}\left\{2q + \frac{\upsilon}{\kappa}\right\}\right|$$

after which owing to dissipation it decreases rapidly. The wave number k increases in the course of time. The new and interesting fact which has been discovered in the course of this research is that the wave number k, corresponding to the time

$$t = \sum_{i=1}^{N}\left(-\frac{1}{2q}\right)\ln\left|\frac{q}{\upsilon k^2(0)}\left\{2q + \frac{\upsilon}{\kappa}\right\}\right| \text{ is}$$

not dependent on the initial conditions and is equal to $k = \sqrt{\frac{-q}{\upsilon}}$.

It should be noted that in each of the cases investigated $q > 0$ corresponds to the situation when the disks are moving towards each other and $q < 0$ to the situation when the disks are moving apart.

Remark 2. Note that if in equation (4.4) $N = 1$, then the results obtained by Craik [2] are retrieved. This case corresponds to a perturbation in a form of one plane wave. The case when $N > 1$ corresponds to plane-wave superposition, which can reduce to the appearance of different space structures.

4.2 Stationary solutions in the form of jets

The solution $\psi = 0$, corresponding to the liquid motion near a stagnation point, has been considered. It is also relevant to find and examine other stationary solutions, such as jets. We consider the flow function in the Riabouchinsky type form:

$$\psi(x,y) = xF(y) + \varphi(y) \qquad (4.5)$$

In this case equation (3.3) becomes

$$\psi_x \omega_y - \psi_y \omega_x = -q(2\omega + y\omega_y + x\omega_x) + \upsilon\,\Delta\omega - \frac{\upsilon}{\kappa}\omega \qquad (4.6)$$

and since

$$\omega = xF''(y) + \varphi''(y) \qquad (4.7)$$

Equation (4.6) can be rewritten as

$$F[xF''' + \varphi'''] - [xF' + \varphi']F'' =$$
$$-q[2xF'' + 2\varphi'' + xyF''' + y\varphi''' + xF''] + \upsilon[xF^{iv} + \varphi^{iv}]$$
$$-\frac{\upsilon}{\kappa}[xF'' + \varphi''] \qquad (4.8)$$

Where a prime denotes a derivative with respect to the argument (here y) and a superscript (iv) denotes the fourth derivative. Equating groups of terms with the same x powers it is possible to obtain the following system:

$$FF''' - F'F'' = q[3F'' + yF'''] - \upsilon F^{iv} + \frac{\upsilon}{\kappa}F'' \qquad (4.9)$$

$$F\varphi''' - \varphi'F'' = q[2\varphi'' + y\varphi'''] - \upsilon\varphi^{iv} + \frac{\upsilon}{\kappa}\varphi'' \qquad (4.10)$$

We consider the particular case when the analytical solution of equation (4.9) is $F = ay$. In this case (4.10) will take the following form:

$$-a\ \varphi'''(y) + q(2\varphi''(y) + y\varphi'''(y)) = \upsilon\varphi^{iv}(y) - \frac{\upsilon}{k}\varphi''(y) \qquad (4.11)$$

After some mathematical transformations and integrating twice, we obtain the following equation.

$$\upsilon\varphi''(y) = (q-a)y\varphi'(y) - \left(2q - \frac{\upsilon}{\kappa}\right)\varphi(y) \qquad (4.12)$$

Which has the form of Hermit's differential equation when two conditions are satisfied: $\frac{q-a}{\nu} = 2$ and $\frac{1}{2\upsilon}\left(2q - \frac{\upsilon}{\kappa}\right)$ is non-negative integer. The solutions of this equation have the following form:

$$\varphi = \frac{d^n}{dy^n}\left(A\exp\left(\frac{1}{2\upsilon(3+n)}\left(2q - \frac{\upsilon}{\kappa}\right)\right)y^2\right) \qquad (4.13)$$

Where the relation between a and q is

$$a = -\frac{1+n}{3+n}q, \quad n \in [0,\infty] \qquad (4.14)$$

Thus the solutions of equation (3.3) can be written as

$$\psi = -\frac{1+n}{3+n}qxy + \frac{d^n}{dx^n}\left(A\exp\left(\left(\frac{1}{2\upsilon(3+n)}\left(2q - \frac{\upsilon}{k}\right)\right)y^2\right)\right) \qquad (4.15)$$

In equation (4.15), the first term denotes the liquid motion corresponding to the potential flow component and the second term denotes (represents) the jet behavior (non-potential flow component). Since $q < 0,\ n > 0,\ \nu > 0,$ it can be seen that this second term approaches zero for $y \to \pm\infty$

5. CONCLUSIONS

In this investigation, we came to know that, if the disks are moving apart ($q \prec 0$) the non-Newtonian fluid is unstable up to certain time $t = \sum_{i=1}^{N}\left(-\frac{1}{2q}\right)\ln\left|\frac{q}{\upsilon k^2(0)}\left\{2q + \frac{\upsilon}{\kappa}\right\}\right|$ then it is stable there after words. This is because of the wave number $k(t)$ increases in the course of time. But in case of the disks are moving towards each other ($q \succ 0$) the non-Newtonian fluid is stable with both the amplitude $A(t)$ and the wave number $k(t)$ at all time.

The solution for the stream function through Hermit's differential equation is given by

$$\psi = -\frac{1+n}{3+n}qxy + \frac{d^n}{dx^n}\left(A\exp\left(\left(\frac{1}{2\upsilon(3+n)}\left(2q - \frac{\upsilon}{k}\right)\right)y^2\right)\right) \qquad (4.11)$$

In this equation the first term denotes the liquid motion corresponding to the potential flow component and the second term denotes the jet behavior.

ACKNOWLEDGMENT

The author Kempe Gowda M. would like to acknowledge the support and encouragement given by the Management, Director and the Principal of Vemana Institute of Technology, Bangalore, India.

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