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A Common Fixed Point Theorem for Weakly Compatible Maps Satisfying Property (E.A.) in Fuzzy Metric Spaces using Strict Contractive Condition

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ABSTRACT

In this paper, employing the property (E.A), we prove a common fixed theorem for weakly compatible maps using strict contractive condition in fuzzy metric space.

Keywords: *Fuzzy metric space, weakly compatible maps, property (E.A.).*

1. INTRODUCTION

In 1986, Jungck [1] introduced the notion of compatible maps for a pair of self mappings. However, the study of common fixed points of non-compatible maps is also very interesting. Aamri and El. Moutawakil [2] generalized the concept of non-compatibility by defining the notion of property (E.A) and proved common fixed point theorems under strict contractive conditions. Jungck and Rhoades [3] initiated the study of weakly compatible maps in metric space and showed that every pair of compatible maps is weakly compatible but reverse is not true. In the literature, many results have been proved for contraction maps satisfying property (E.A.) in different settings such as probabilistic metric spaces [4, 5]; fuzzy metric spaces [6, 7].

In this paper, employing the property (E.A), we prove a common fixed theorem for weakly compatible maps using strict contractive condition in fuzzy metric space.

2. PRELIMINARIES

Definition 2.1: [8]

A binary operation * : [0,1]×[0,1] $\longrightarrow$ [0,1] is continuous t-norm if * satisfies the following conditions:

(i) * is commutative and associative;

(ii) * is continuous;

(iii) A * 1 = a for all $a \in [0,1]$;

(iv) A * b ≤ c * d whenever a ≤ c and b ≤ d for all $a,b,c,d \in [0,1]$.

Kramosil I and Michalek J.[9] introduced the concept of fuzzy metric spaces as follows:

Definition 2.2: [9]

The 3-tuple (X, M, *) is called a fuzzy metric space (shortly, FM-space) if X is an arbitrary set, * is a continuous t-norm and M is a fuzzy set in $X^2 \times [0, \infty)$ satisfying the following conditions:

(FM-1) M(x, y, 0) = 0,

(FM-2) M(x, y, t) = 1, for all $t > 0$ if and only if x = y,

(FM-3) M(x, y, t) = M(y, x. t),

(FM-4) M(x, y, t)* M(y, z, s) ≤ M(x, z, t + s) (Triangular inequality)

(FM-5) M(x, y,.) : [0, 1) $\rightarrow$ [0, 1] is left continuous for all $x, y, z \in X$ and $s, t > 0$.

Note that M(x, y, t) can be thought of as the degree of nearness between x and y with respect to t.
We can fuzzily examples of metric spaces into fuzzy metric spaces in a natural way:
Let (X, d) be a metric space. Define

a * b = a + b for all a, b in X.

Define $M(x, y, t) = \dfrac{t}{t + d(x, y)}$ for all x, y in X and $t > 0$. Then (X,M, *) is a fuzzy metric space and this fuzzy metric induced by a metric d is called the **Standard fuzzy metric**.

Consider M to be a fuzzy metric space with the following condition:

(FM-6) $\lim_{t \to \infty} M(x, y, t) = 1$ for all x, y in X and $t > 0$.

Definition 2.3: [9]

Let $(X, M, *)$ be fuzzy metric space. Then

(a) a sequence $\{x_n\}$ in X is said to be Cauchy sequence if, for all $t > 0$ and $p > 0$, $\lim_{n \to \infty} M(x_{n+p}, x_n, t) = 1$ and

(b) a sequence $\{x_n\}$ in X is said to be convergent to a point $x \in X$ if, for all $t > 0$,

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$\lim_{n\to\infty} M(x_n, x, t) = 1$.

Definition 2.4: [9]

A fuzzy metric space $(X, M, *)$ is said to be complete if and only if every Cauchy sequence in X is convergent.

Example 2.1: [9]

Let $X = \{1/n : n \in N\} \cup \{0\}$ and let * be the continuous t-norm and defined by $a*b = ab$ for all $a, b \in [0,1]$. For each $t \in (0, \infty)$ and $x, y \in X$, define M by

$$M(x,y,t) = \begin{cases} \frac{t}{t+|x-y|}, & t > 0, \\ 0 & t = 0 \end{cases}$$

Clearly, $(X, M, *)$ is complete fuzzy metric space?

Definition 2.5

A pair of self mappings (S, T) of a fuzzy metric space $(X, M, *)$ is said to be commuting if M(STx ,TSx , t) = 1 for all $x \in X$.

Definition 2.6

A pair of self mappings (S, T) of a fuzzy metric space $(X, M, *)$ is said to be weakly commuting if
M (STx , TSx , t) $\geq$ M(Sx, Tx, t)
for all $x \in X$ and $t > 0$.

Definition 2.7

A pair of self mappings (S, T) of a fuzzy metric space $(X, M, *)$ is said to be compatible if $\lim_{n\to\infty} M(STx_n, TSx_n, t) = 1$ for all $t > 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = u$ for some u in X.

Definition 2.8

Let $(X, M, *)$ be a fuzzy metric space. S and T be self maps on X. A point x in X is called a coincidence point of S and T iff Sx = Tx. In this case, w = Sx = Tx is called a point of coincidence of S and T.

Definition 2.9

A pair of self mappings (S, T) of a fuzzy metric space $(X, M, *)$ is said to be weakly compatible if they commute at the coincidence points i.e., if Su = Tu for some $u \in X$, then STu =TSu.

It is easy to see that two compatible maps are weakly compatible but converse is not true.

Definition 2.10: [2]

A pair of self mappings (S, T) of a fuzzy metric space $(X, M, *)$ is said to satisfy the property (E.A) if there exist a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z$ for some $z \in X$.

Example 2.2: [2]

Let X = [0, ∞). Consider $(X, M, *)$ be a fuzzy metric space as in Example 2.1. Define S, T: X →X by Sx = $\frac{x}{5}$ and Tx = $\frac{2x}{5}$ for all $x \in X$. Clearly, for sequence $\{x_n\} = \{1/n\}$, S and T satisfy property (E.A).

3. MAIN RESULTS

Theorem 3.1

Let $(X, M, *)$ be a fuzzy metric space with a * b = min {a, b} for all a, b in [0, 1] and S and T be weakly compatible mappings of X into itself such that :

(3.1) the pair (S, T) satisfies the property (E.A.);

(3.2) for any x, y in X and for all $t > 0$ there exists $k \in (0,1)$ such that,

$$M^2(Tu, Tv, kt) \geq \left[\min \left\{ \begin{array}{l} M(Su, Sv, t), \\ M(Su, Tu, t), \\ M(Sv, Tv, t), \\ M(Sv, Tu, t), \\ M(Su, Tv, t) \end{array} \right\} \right]^2$$

If S(X) be a closed subset of X, then S and T have a unique common fixed point.

Proof:

As the pair (S, T) satisfies property (E.A.), then there exist a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z$ for some $z \in X$. Since S(X)

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is closed subset of X, z = Sa for some a in X. Therefore, $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z = Sa.$.

By (3.1), take $u = x_n, v = a$, we get

$$M^2(Tx_n, Ta, kt) \ge \left[\min \left\{ \begin{array}{l} M(Sx_n, Sa, t), \\ M(Sx_n, Tx_n, t), \\ M(Sa, Ta, t), \\ M(Sa, Tx_n, t), \\ M(Sx_n, Ta, t) \end{array} \right\} \right]^2$$

$n \to \infty$

$$M^2(Sa, Ta, kt) \ge \left[\min \left\{ \begin{array}{l} M(Sa, Sa, t), \\ M(z, z, t), \\ M(Sa, Ta, t), \\ M(Sa, Sa, t), \\ M(Sa, Ta, t) \end{array} \right\} \right]^2$$

$$M^2(Sa, Ta, kt) \ge M^2(Sa, Ta, t)$$
$$\ge M^2(Sa, Ta, kt)$$
$$M^2(Sa, Ta, kt) = M^2(Sa, Ta, t)$$

for all $x \in [kt, t]$

$$M^2(Sa, Ta, t) = M^2(Sa, Ta, t/k)$$

for all $x \in [t, t/k]$.

Similarly,

$$M^2(Sa, Ta, t) = M^2(Sa, Ta, t/k^2)$$

for all $x \in [t/k, t/k^2]$.

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Continuing like this, we can prove that

$$M^2(Sa, Ta, t) = M^2(Sa, Ta, t/k^n)$$

for all $x \in [t/k^{n-1}, t/k^n]$.

Take $n \to \infty$

$M^2(Sa, Ta, t) = 1$, hence, Sa = Ta. As S and T are weakly compatible, STa = TSa = TTa = SSa. Now, we claim that Ta is common fixed point of S and T.

By (3.1), take $u = a, v = Ta$, we get

$$M^2(Ta, TTa, kt) \ge \left[\min \left\{ \begin{array}{l} M(Sa, STa, t), \\ M(Sa, Ta, t), \\ M(STa, TTa, t), \\ M(STa, Ta, t), \\ M(Sa, TTa, t) \end{array} \right\} \right]^2$$

$$M^2(Ta, TTa, kt) \ge \left[\min \left\{ \begin{array}{l} M(Ta, TTa, t), 1, 1, \\ M(TTa, Ta, t), \\ M(Ta, TTa, t) \end{array} \right\} \right]^2$$

$$M^2(Ta, TTa, kt) \ge M^2(Ta, TTa, t)$$
$$\ge M^2(Ta, TTa, kt)$$
$$M^2(Ta, TTa, kt) = M^2(Ta, TTa, t)$$

for all $x \in [kt, t]$.

By applying same steps as above, one can easily show that

$$M^2(Ta, TTa, t) = 1.$$

This gives, Ta = TTa. Therefore, STa = TTa = Ta. Hence, S and T has a common fixed point.

For uniqueness, let w and z be two fixed points of S and T, then by (3.1), take u = w, v = z, we get

$$M^2(Tw, Tz, kt) \ge \left[\min \left\{ \begin{array}{l} M(Sw, Sz, t), \\ M(Sw, Tw, t), \\ M(Sz, Tz, t), \\ M(Sz, Tw, t), \\ M(Sw, Tz, t) \end{array} \right\} \right]^2$$

$$M^2(w, z, kt) \ge \left[\min \left\{ \begin{array}{l} M(w, z, t), \\ M(w, w, t), \\ M(z, z, t), \\ M(z, w, t), \\ M(w, z, t) \end{array} \right\} \right]^2$$

$$M^2(w, z, kt) \ge M^2(w, z, t)$$
$$\ge M^2(w, z, kt)$$

this gives,

$$M^2(w, z, kt) = M^2(w, z, t)$$

for all $x \in [kt, t]$.

Continuing in a same manner as above, one can easily show that w = z. Hence, z is unique common fixed point of S and T.

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