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A Fixed Point Theorem for Surfaces

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ABSTRACT

The article of this note is to outline a proof of "every homeomorphism of the plane or surface into itself that leaves a continuum M$\subset R^2$ invariant has a fixed point in F (M)". We prove that if F is a Contraction Mapping, then there is at least one fixed point of F, where $M \subset E^2$ is a compact surface and $F: M \to M$ is a surface mapping

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1. INTRODUCTION

A homeomorphism $f: X \to X$ of a compact metrizable space X is called point wise recurrent if $X \in L^+(x) \cap L^-(x)$ for every $x \in X$, where

$$L^+(x) = \{y \in X: f^{nk}(x) \to y \text{ for some } nk \to +\infty\}$$

is the positive limit set of x with respect to f and $L^-(x)$ is the positive limit set of x with respect to f^{-1}. A point wise recurrent, orientation preserving homeomorphism of E is topologically conjugate to a rotation. This is not true for point wise recurrent, orientation preserving homeomorphisms of the 2-sphere E^2 and it is an interesting problem to seek for additional conditions which ensure topological conjugacy to a rotation.

A first step towards a characterization of rotations modulo topological conjugacy in the class of point wise recurrent, orientation preserving homeomorphisms of E^2 would be a theorem which guarantees the existence of only two fixed points. (Athanassopoulos 2006)

A weakly almost periodic homeomorphism of a compact metrizable space is point wise recurrent. It is proved in W.K. Mason's paper that a weakly almost periodic, orientation preserving homeomorphism of E^2, different from the identity, has exactly two fixed points. In this note we generalize this result to the class of point wise recurrent homeomorphisms with stable fixed points.

More precisely, we prove that if f : $E^2 \subset R^2 \to E^2$ is a point wise recurrent, orientation preserving homeomorphism, different from the identity, and if every fixed point of f is stable, then f must have exactly two fixed points. A compact invariant set of f is stable, if it has a neighborhood basis consisting of f -invariant, open sets. If f is weakly almost periodic, then every orbit closure of f is stable. The idea of proof was inspired by the proof for weakly almost periodic homeomorphisms in (Mason 1973), but is considerably simpler and shorter. This is due to the fact we prove first, that a stable fixed point of a point wise recurrent, orientation preserving homeomorphism f of (E^2) has a neighborhood basis consisting of f -invariant topological open discs (Theorem 1.4).

This permits us to use the Brouwer Translation Theorem instead of the theory of prime ends, as it is done in W.K. Mason's paper.

Although point wise recurrence is a property which is inherited by the iterates of a homeomorphism of a metric space, the stability of fixed points is not. It is clear however that if f: $E^2 \to E^2$ is a point wise recurrent, orientation preserving homeomorphism, different from the identity, which has stable fixed points and has no periodic point, other than fixed, then f^n has the same properties for $n \neq 0$.

As an application of the main theorem, we show that every stable minimal set of a homeomorphism in this class is connected and its complement in E^2 has exactly two connected components, which generalizes in W.K. Mason's paper theorem 6.

In the final section we are concerned with the problem of whether a lifting to the universal covering space R^2 of the restriction of a point wise recurrent, orientation preserving homeomorphism f of (E^2) which is different from the identity and has stable fixed points, to the complement of the fixed point set, is topologically conjugate to translation. This is closely related to a conjecture made by Winkelnkemper (1988).

He give a partial affirmative answer in case f is a C^1 diffeomorphism near the fixed points, under the assumption

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that the infinitesimal rotation numbers at the fixed points are non-zero (Mason 1973). All sets will be assumed to be subsets of the plane unless otherwise indicated.

Definition 1.1

If A is a bounded set then F(A) is the smallest compact set that contains A and has a connected complement.

Definition 1.2

$Let\ f: E^2 \to E^2$ Be a homeomorphism of the plane that admits a Lyapunov metric function d. A point $x \in E^2$ is a stable (unstable) point if given any $\tau' > 0$ there exists $\tau > 0$ such that for every $y \in Bk(x)$, it follows that
d(f^n(x), f^n(y)) $> \tau'$
for each $n \geq 0$(Groisman 2007)

Lemma 1.1

If $f: E^2 = R^2 \to E^2$is an orientation preserving, point wise recurrent homeomorphism, different from the identity, then the fixed point set Fix (f) of f is not connected and no connected component of E^2 \ Fix (f) is topologically an open disc(Athanassopoulos 2006).

Theorem 1.1

Let f: U →E^2 be a map defined on a simple closed curve U. If there is a partition of U, {x_0, x_1, x_2, … , $x_n = x_0$} and arcs A_1, A_2, A_3, … , A_n an F(U) such that Ai joins f(x_{i-1}) to f(x_i) and

$$x_{i-1}x_i \cap F(f[x_{i-1}\ x_i] \cup A_i) = \emptyset,$$

then every extension of f to a map defined on F(U) has a fixed point(Bell 1976).

Theorem 1.2

There does not exist a homeomorphism of the plane into itself that leaves M invariant and has the property that small arcs that cut across the out channel have images that cut across the out channel further out and small arcs that cut across the in channel have images that cut across the in channel further in Bell's paper (1976).

Theorem 1.3

Let U be a metrich space. Suppose that T is continuous mapping of (a closed subset of) U in to acompact subset of U and that, for each $\varepsilon > 0$,there exist $x(\varepsilon)$ such that

$$d(Tx(\varepsilon), x(\varepsilon)) < \varepsilon$$

Then T has a fixed point.(Smart 1974).

Definition 1.3

Let T be a mapping of a metric M into M. We say that T is a contraction mapping if there exists number k such that $0 < k < 1$ and

$$d(Tx, Ty) \leq kd(x, y) \quad (\forall x, y \in M)$$

The following result is called the Contraction Mapping Theorem.(Smart 1974).

Definition 1.3

A function $F: M \to N$ from one surface to another is differentiable provided that for each patch $X\ \text{in}\ M\ \text{and}\ Y\ \text{in}\ N$ the composite function $Y^{-1}FX$ is Euclidian differentiable (and difinedon an open set of R^2).F is then called a mapping of surfaces(O'Neill 1997).(see Figure 1.1)

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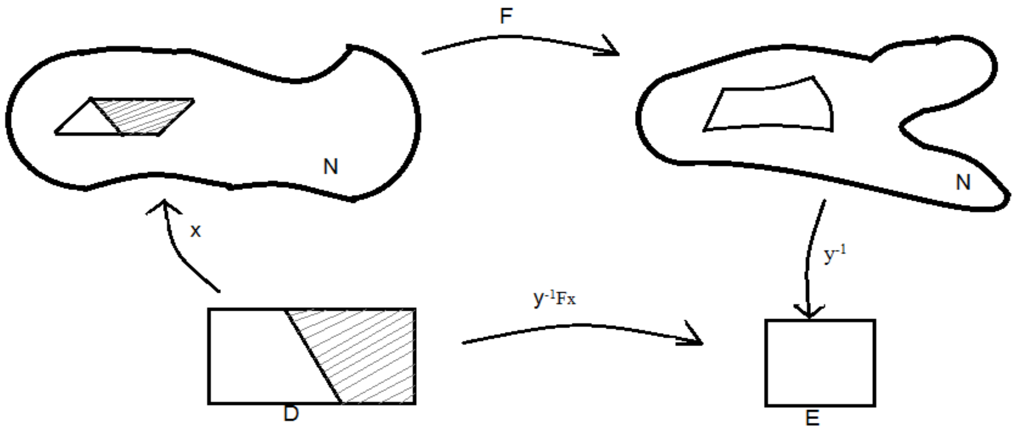

Fig 1.1

Lemma 1.2

(M.W. Hirsch). If $f: E^2 \subset R^2 \to E^2$ is an orientation preserving, point wise recurrent homeomorphism, different from the identity, then Fix (f) has at least two acyclic connected components(Athanassopoulos 2006).

Theorem 1.4

Let f: $E^2 \to E^2$ be an orientation preserving, point wise recurrent homeo-morphism. *If* $U \subset E^2$ is an f -stable, f -invariant, acyclic continuum, then every neighborhood of U contains an f - invariant open neighborhood of U which is topologically an open disc(Athanassopoulos 2006).

Theorem 1.5

(Principle of successive approximations) if F *is* continuous on a Hausdorf topological space M *to* M and if $\lim F^n x = y$ *exist then* $Fy = y$.

Proof.

$Fy = (\lim F^n x) = \lim F^{n+1} x = y$. (Smart 1974).

2. MAİN RESULT

Main Theorem

Let $M \subset E^3$ be a compact surface and $F: M \to M$ be a surface mapping. If F is a Contraction Mapping, then there is at least one fixed point of F.

Proof

Let the mapping F satisfy definition 1.3 for every $k < 1$. Choose any point x in M.The sequence of points $F^n x$ satisfies, for $n > 0$,

$$d(F^n x, F^{n+1} x) \le kd(F^{n-1} x, F^n x),$$

so that by induction

$$(F^n x, F^{n+1} x) \le k^n d(x, Fx).$$

By the triangle inequality we have for $m \ge n$

$$d(F^n x, F^m x) \le d(F^n x, F^{n+1} x) + d(F^{n+1} x, F^{n+2} x) + \dots + d(F^{m-1} x, F^m x) \le k^n + k^{n+1} + \dots + k^{m-1})d(x, Fx)$$
$$\le k^n (1-k)^{-1} d(x, Fx)$$

Thus $d(F^n x, F^m x) \to 0$ if $m, n \to 0$. Since M is complete the sequence $F^n x$ *has a* limit y *in* M. By Theorem 1.5, y is a fixed point for F.

This fixed point is unique since if $Fy = y$ any $Fz = z$, we have

$$d(y, z) = d(Fy, Fz) \le kd(y, z)$$

So that $d(y, z) = 0$; that is , $y = z$.

Example 2.1

Let be $M \subset E^3$, $F: M \to M$ and $F(x_1, x_2, x_3) = (\frac{1}{2}x_1 - 1, \frac{1}{2}x_2 + 1, \frac{1}{2}x_3 - 3)$. Then F is a contraction mapping and fixed point of F is $(-2, 2, -6)$.

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Solution. Let

be $x, y \in M$ and $x = (x_1, x_2, x_3), y = (y_1, y_2, y_3)$.

$$d(Fx, Fy) = \|Fx - Fy\|$$

$$= \sqrt{(\tfrac{1}{2}x_1 - 1 - \tfrac{1}{2}y_1 + 1)^2 + (\tfrac{1}{2}x_2 + 1 - \tfrac{1}{2}y_2 - 1)^2}$$

$$= \sqrt{(\tfrac{1}{2}x_1 - \tfrac{1}{2}y_1)^2 + (\tfrac{1}{2}x_2 - \tfrac{1}{2}y_2)^2 + (\tfrac{1}{2}x_3 - \tfrac{1}{2}y_3)^2}$$

$$= {}^1\!/_2 \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2}$$

$$\leq {}^1\!/_2\, d(x, y))$$

$k = {}^1\!/_2$ so F is a contraction mapping.

Also fixed point of F is

$$d(F^2x, F^2y) = \|F^2x - F^2y\|$$

$$= {}^1\!/_4 \|x - y\|$$

$$d(F^3x, F^3y) = \|F^3x - F^3y\|$$

$$- {}^1\!/_3 \|x - y\|$$

:

$$d(F^n x, F^n y) = \|F^n x - F^n y\|$$

$$= {}^1\!/_{2^n} \|x - y\|$$

$\leq \lim_{n\to\infty} {}^1\!/_{2^n} \|x - y\| \to 0$ (şekil 3.1)

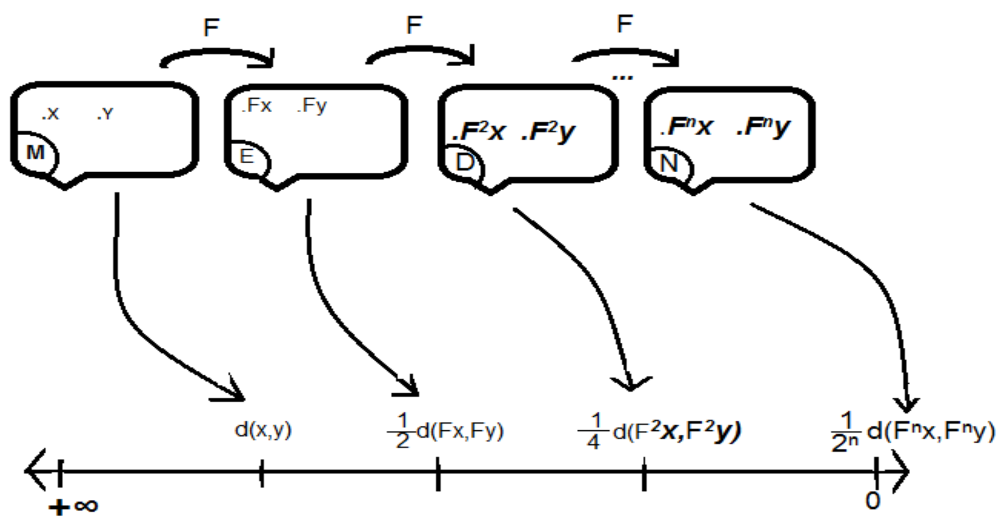

Fig 2.1

Also; $F(x_1, x_2, x_3) = (\tfrac{1}{2}x_1 - 1, \tfrac{1}{2}x_2 + 1, \tfrac{1}{2}x_3 - 3) = (x_1, x_2, x_3)$

$$\frac{1}{2}x_1 - 1 = x_1 \rightarrow x_1 = -2$$

$$\frac{1}{2}x_2 + 1 = x_2 \rightarrow x_2 = 2$$

$$\frac{1}{2}x_3 - 3 = x_3 \rightarrow x_3 = -6$$

Then, F is a contraction mapping and fixed point of F is $(-2, 2, -6)$.

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